Basic Category Theory

Jaap van Oosten

Jaap van Oosten
Department of Mathematics
Utrecht University
The Netherlands
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1 Categories and Functors

1.1 Definitions and examples

A *category* $\mathcal{C}$ is given by a collection $\mathcal{C}_0$ of *objects* and a collection $\mathcal{C}_1$ of *arrows* which have the following structure.

- Each arrow has a *domain* and a *codomain* which are objects; one writes $f : X \to Y$ or $X \xrightarrow{f} Y$ if X is the domain of the arrow f, and Y its codomain. One also writes $X = \mathrm{dom}(f)$ and $Y = \mathrm{cod}(f)$;
- Given two arrows f and g such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, the *composition* of f and g, written gf, is defined and has domain $\mathrm{dom}(f)$ and codomain $\mathrm{cod}(g)$:
$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \;\mapsto\; (X \xrightarrow{gf} Z)$$
- Composition is *associative*, that is: given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, $h(gf) = (hg)f$;
- For every object X there is an *identity* arrow $\mathrm{id}_X : X \to X$, satisfying $\mathrm{id}_X g = g$ for every $g : Y \to X$ and $f\mathrm{id}_X = f$ for every $f : X \to Y$.

Exercise 1 Show that id_X is the *unique* arrow with domain X and codomain X with this property.

Instead of “arrow” we also use the terms “morphism” or “map”.

Examples

a) **1** is the category with one object $*$ and one arrow, id_*;

b) **0** is the empty category. It has no objects and no arrows.

c) A *preorder* is a set X together with a binary relation $\leq$ which is reflexive (i.e. $x \leq x$ for all $x \in X$) and transitive (i.e. $x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in X$). This can be viewed as a category, with set of objects X and for every pair of objects (x, y) such that $x \leq y$, exactly one arrow: $x \to y$.

Exercise 2 Prove this. Prove also the converse: if $\mathcal{C}$ is a category such that $\mathcal{C}_0$ is a set, and such that for any two objects X, Y of $\mathcal{C}$ there is at most one arrow: $X \to Y$, then $\mathcal{C}_0$ is a preordered set.

d) A *monoid* is a set X together with a binary operation, written like multiplication: xy for $x, y \in X$, which is associative and has a unit element $e \in X$, satisfying $ex = xe = x$ for all $x \in X$. Such a monoid is a category with one object, and an arrow x for every $x \in X$.

e) Set is the category which has the class of all sets as objects, and functions between sets as arrows.

Most basic categories have as objects certain mathematical structures, and the structure-preserving functions as morphisms. Examples:

f) Top is the category of topological spaces and continuous functions.

g) Grp is the category of groups and group homomorphisms.

h) Rng is the category of rings and ring homomorphisms.

i) Grph is the category of graphs and graph homomorphisms.

j) Pos is the category of partially ordered sets and monotone functions.

Given two categories C and D, a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of operations $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$, such that for each $f : X \to Y$, $F_1(f) : F_0(X) \to F_0(Y)$ and:

- for $X \xrightarrow{f} Y \xrightarrow{g} Z$, $F_1(gf) = F_1(g)F_1(f)$;
- $F_1(\mathrm{id}_X) = \mathrm{id}_{F_0(X)}$ for each $X \in \mathcal{C}_0$.

But usually we write just F instead of F_0, F_1.

Examples.

a) There is a functor U : Top $\to$ Set which assigns to any topological space X its underlying set. We call this functor "forgetful": it "forgets" the mathematical structure. Similarly, there are forgetful functors Grp $\to$ Set, Grph $\to$ Set, Rng $\to$ Set, Pos $\to$ Set etcetera;

b) For every category C there is a unique functor $\mathcal{C} \to \mathbf{1}$ and a unique one $\mathbf{0} \to \mathcal{C}$;

c) Given two categories C and D we can define the *product category* $\mathcal{C} \times \mathcal{D}$ which has as objects pairs $(C, D) \in \mathcal{C}_0 \times \mathcal{D}_0$, and as arrows:$(C, D) \to (C', D')$ pairs (f, g) with $f : C \to C'$ in C, and $g : D \to D'$ in D. There are functors $\pi_0 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$;

d) Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ one can define the composition $GF : \mathcal{C} \to \mathcal{E}$. This composition is of course associative and since we have, for any category $\mathcal{C}$, the *identity functor* $\mathcal{C} \to \mathcal{C}$, we have a category Cat which has categories as objects and functors as morphisms.

e) Given a set A, consider the set $\tilde{A}$ of strings $a_1 \dots a_n$ on the alphabet $A \cup A^{-1}$ (A^{-1} consists of elements a^{-1} for each element a of A; the sets A and A^{-1} are disjoint and in 1-1 correspondence with each other), such that for no $x \in A$, xx^{-1} or $x^{-1}x$ is a substring of $a_1 \dots a_n$. Given two such strings $\vec{a} = a_1 \dots a_n, \vec{b} = b_1 \dots b_m$, let $\vec{a} \star \vec{b}$ the string formed by first taking $a_1 \dots a_n b_1 \dots b_m$ and then removing from this string, successively, substrings of form xx^{-1} or $x^{-1}x$, until one has an element of $\tilde{A}$.

   This defines a group structure on $\tilde{A}$. $\tilde{A}$ is called the *free group* on the set A.

Exercise 3 Prove this, and prove that the assignment $A \mapsto \tilde{A}$ is part of a functor: Set $\to$ Grp. This functor is called the *free functor.*

f) Every directed graph can be made into a category as follows: the objects are the vertices of the graph and the arrows are paths in the graph. This defines a functor from the category Dgrph of directed graphs to Cat. The image of a directed graph D under this functor is called the category *generated* by the graph D.

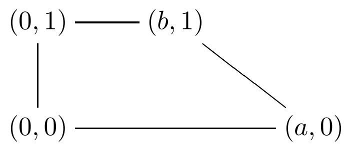
g) **Quotient categories**. Given a category C, a *congruence relation* on $\mathcal{C}$ specifies, for each pair of objects X, Y, an equivalence relation $\sim_{X,Y}$ on the class of arrows $\mathcal{C}(X, Y)$ which have domain X and codomain Y, such that

- for $f, g : X \to Y$ and $h : Y \to Z$, if $f \sim_{X,Y} g$ then $hf \sim_{X,Z} hg$;
- for $f : X \to Y$ and $g, h : Y \to Z$, if $g \sim_{Y,Z} h$ then $gf \sim_{X,Z} hf$.

Given such a congruence relation $\sim$ on $\mathcal{C}$, one can form the quotient category $\mathcal{C}/\sim$ which has the same objects as $\mathcal{C}$, and arrows $X \to Y$ are $\sim_{X,Y}$-equivalence classes of arrows $X \to Y$ in C.

Exercise 4 Show this and show that there is a functor $\mathcal{C} \to \mathcal{C}/\sim$, which takes each arrow of $\mathcal{C}$ to its equivalence class.

h) An example of this is the following ("homotopy"). Given a topological space X and points $x, y \in X$, a *path* from x to y is a continuous mapping f from some closed interval $[0, a]$ to X with $f(0) = x$ and $f(a) = y$. If $f : [0, a] \to X$ is a path from x to y and $g : [0, b] \to X$ is a path from y to z there is a path $gf : [0, a+b] \to X$ (defined by $gf(t) = \begin{cases} f(t) & t \leq a \\ g(t-a) & \text{else} \end{cases}$) from x to z. This makes X into a category, the *path category* of X, and of course this also defines a functor Top $\to$ Cat. Now given paths $f : [0, a] \to X$, $g : [0, b] \to X$, both from x to y, one can define $f \sim_{x,y} g$ if there is a continuous map $F : A \to X$ where A is the area:



in $\mathbb{R}^2$, such that

$$
\begin{array}{lll}
F(t, 0) = & f(t) & \\
F(t, 1) = & g(t) & \\
F(0, s) = & x & s \in [0, 1] \\
F(s, t) = & y & (s, t) \text{ on the segment } (b, 1) - (a, 0)
\end{array}
$$

One can easily show that this is a congruence relation. The quotient of the path category by this congruence relation is a category called the category of *homotopy classes* of paths in X.

i) let $\mathcal{C}$ be a category such that for every pair (X, Y) of objects the class $\mathcal{C}(X, Y)$ of arrows from X to Y is a set (such $\mathcal{C}$ is called *locally small*).

For any object C of $\mathcal{C}$ then, there is a functor $h_C : \mathcal{C} \to \mathrm{Set}$ which assigns to any object C' the set $\mathcal{C}(C, C')$. Any arrow $f : C' \to C''$ gives by composition a function $\mathcal{C}(C, C') \to \mathcal{C}(C, C'')$, so we have a functor. A functor of this form is called a *representable functor*.

j) Let $\mathcal{C}$ be a category and C an object of $\mathcal{C}$. The *slice category* $\mathcal{C}/C$ has as objects all arrows g which have codomain C. An arrow from $g : D \to C$ to $h : E \to C$ in $\mathcal{C}/C$ is an arrow $k : D \to E$ in $\mathcal{C}$ such that $hk = g$. Draw like:

$$\begin{tikzcd} D \arrow[rr, "k"] \arrow[dr, "g"'] & & E \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

We say that *this diagram commutes* if we mean that $hk = g$.

Exercise 5 Convince yourself that the assignment $C \mapsto \mathcal{C}/C$ gives rise to a functor $\mathcal{C} \to \mathrm{Cat}$.

k) Recall that for every group $(G, \cdot)$ we can form a group $(G, \star)$ by putting $f \star g = g \cdot f$.

For categories the same construction is available: given $\mathcal{C}$ we can form a category $\mathcal{C}^{\mathrm{op}}$ which has the same objects and arrows as C, but with reversed direction; so if $f : X \to Y$ in C then $f : Y \to X$ in $\mathcal{C}^{\mathrm{op}}$. To make it notationally clear, write $\bar{f}$ for the arrow $Y \to X$ corresponding to $f : X \to Y$ in C. Composition in $\mathcal{C}^{\mathrm{op}}$ is defined by:

$$\bar{f}\bar{g} = \overline{gf}$$

Often one reads the term "contravariant functor" in the literature. What I call functor, is then called "covariant functor". A contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ inverts the direction of the arrows, so $F_1(f) : F_0(\mathrm{cod}(f)) \to F_0(\mathrm{dom}(f))$ for arrows f in $\mathcal{C}$. In other words, a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ (equivalently, from $\mathcal{C}$ to $\mathcal{D}^{\mathrm{op}}$).

Of course, any functor $F : \mathcal{C} \to \mathcal{D}$ gives a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. In fact, we have a functor $(-)^{\mathrm{op}} : \mathrm{Cat} \to \mathrm{Cat}$.

Exercise 6 Let $\mathcal{C}$ be locally small. Show that there is a functor (the "Hom functor") $\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$, assigning to the pair (A, B) of objects of $\mathcal{C}$, the set $\mathcal{C}(A, B)$.

l) Given a partially ordered set $(X, \leq)$ we make a topological space by defining $U \subseteq X$ to be open iff for all $x, y \in X$, $x \leq y$ and $x \in U$ imply $y \in U$ (U is "upwards closed", or an "upper set"). This is a topology, called the *Alexandroff topology* w.r.t. the order $\leq$.

If $(X, \leq)$ and $(Y, \leq)$ are two partially ordered sets, a function $f : X \to Y$ is monotone for the orderings if and only if f is continuous for the Alexandroff topologies. This gives an important functor: Pos $\to$ Top.

Exercise 7 Show that the construction of the quotient category in example g) generalizes that of a quotient group by a normal subgroup. That is, regard a group G as a category with one object; show that there is a bijection between congruence relations on G and normal subgroups of G, and that for a normal subgroup N of G, the quotient category by the congruence relation corresponding to N, is to the quotient group G/N.

m) "Abelianization". Let Abgp be the category of abelian groups and homomorphisms. For every group G the subgroup $[G, G]$ generated by all elements of form $aba^{-1}b^{-1}$ is a normal subgroup. $G/[G, G]$ is abelian, and for every group homomorphism $\phi : G \to H$ with H abelian, there is a unique homomorphism $\bar{\phi} : G/[G, G] \to H$ such that the diagram

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "\phi"] & \\ G/[G,G] \arrow[rr, "\bar{\phi}"'] & & H \end{tikzcd}$$

commutes. Show that this gives a functor: Grp $\to$ Abgp.

n) "Specialization ordering". Given a topological space X, you can define a preorder $\leq_s$ on X as follows: say $x \leq_s y$ if for all open sets U, if $x \in U$ then $y \in U$. $\leq_s$ is a partial order iff X is a T_0-space.

For many spaces, $\leq_s$ is trivial (in particular when X is T_1) but in case X is for example the Alexandroff topology on a poset $(X, \leq)$ as in l), then $x \leq_s y$ iff $x \leq y$.

Exercise 8 If $f : X \to Y$ is a continuous map of topological spaces then f is monotone w.r.t. the specialization orderings $\leq_s$. This defines a functor Top $\to$ Preord, where Preord is the category of preorders and monotone functions.

Exercise 9 Let X be the category defined as follows: objects are pairs (I, x) where I is an open interval in $\mathbb{R}$ and $x \in I$. Morphisms $(I, x) \to (J, y)$ are differentiable functions $f : I \to J$ such that $f(x) = y$.

Let Y be the (multiplicative) monoid $\mathbb{R}$, considered as a category. Show that the operation which sends an arrow $f : (I, x) \to (J, y)$ to $f'(x)$, determines a functor $X \to Y$. On which basic fact of elementary Calculus does this rely?

1.2 Some special objects and arrows

We call an arrow $f : A \to B$ *mono* (or a monomorphism, or monomorphic) in a category $\mathcal{C}$, if for any other object C and for any pair of arrows $g, h : C \to A$, $fg = fh$ implies $g = h$.

In Set, f is mono iff f is an injective function. The same is true for Grp, Grph, Rng, Preord, Pos,...

We call an arrow $f : A \to B$ *epi* (epimorphism, epimorphic) if for any pair $g, h : B \to C$, $gf = hf$ implies $g = h$.

The definition of epi is "dual" to the definition of mono. That is, f is epi in the category $\mathcal{C}$ if and only if f is mono in $\mathcal{C}^{\mathrm{op}}$, and vice versa. In general, given a property P of an object, arrow, diagram,... we can associate with P the dual property P^{op}: the object or arrow has property P^{op} in $\mathcal{C}$ iff it has P in $\mathcal{C}^{\mathrm{op}}$.

The *duality principle*, a very important, albeit trivial, principle in category theory, says that any valid statement about categories, involving the properties $P_1, \ldots, P_n$ implies the "dualized" statement (where direction of arrows is reversed) with the P_i replaced by P_i^{op}.

Example. If gf is mono, then f is mono. From this, "by duality", if fg is epi, then f is epi.

Exercise 10 Prove these statements.

In Set, f is epi iff f is a surjective function. This holds (less trivially!) also for Grp, but not for Mon, the category of monoids and monoid homomorphisms:

Example. In Mon, the embedding $\mathbb{N} \to \mathbb{Z}$ is an epimorphism.

For, suppose $\mathbb{Z} \underset{g}{\overset{f}{\rightrightarrows}} (M, e, \star)$ two monoid homomorphisms which agree on the nonnegative integers. Then

$$f(-1) = f(-1) \star g(1) \star g(-1) = f(-1) \star f(1) \star g(-1) = g(-1)$$

so f and g agree on the whole of $\mathbb{Z}$.

We say a functor F *preserves* a property P if whenever an object or arrow (or...) has P, its F-image does so.

Now a functor does not in general preserve monos or epis: the example of Mon shows that the forgetful functor Mon $\to$ Set does not preserve epis.

An epi $f : A \to B$ is called *split* if there is $g : B \to A$ such that $fg = \mathrm{id}_B$ (other names: in this case g is called a *section* of f, and f a *retraction* of g).

Exercise 11 By duality, define what a split mono is. Prove that every functor preserves split epis and monos.

A morphism $f : A \to B$ is an *isomorphism* if there is $g : B \to A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$. We call g the *inverse* of f (and vice versa, of course); it is unique if it exists. We also write $g = f^{-1}$.

Every functor preserves isomorphisms.

Exercise 12 In Set, every arrow which is both epi and mono is an isomorphism. Not so in Mon, as we have seen. Here's another one: let CRng1 be the category of commutative rings with 1, and ring homomorphisms (preserving 1) as arrows. Show that the embedding $\mathbb{Z} \to \mathbb{Q}$ is epi in CRng1.

Exercise 13 i) If two of f, g and fg are iso, then so is the third;

ii) if f is epi and split mono, it is iso;

iii) if f is split epi and mono, f is iso.

A functor F *reflects* a property P if whenever the F-image of something (object, arrow,...) has P, then that something has.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if for every two objects A, B of $\mathcal{C}$, $F : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is a surjection. F is *faithful* if this map is always injective.

Exercise 14 A faithful functor reflects epis and monos.

An object X is called *terminal* if for any object Y there is exactly one morphism $Y \to X$ in the category. Dually, X is *initial* if for all Y there is exactly one $X \to Y$.

Exercise 15 A full and faithful functor reflects the property of being a terminal (or initial) object.

Exercise 16 If X and X' are two terminal objects, they are *isomorphic*, that is there exists an isomorphism between them. Same for initial objects.

Exercise 17 Let $\sim$ be a congruence on the category $\mathcal{C}$, as in example g). Show: if f and g are arrows $X \to Y$ with inverses f^{-1} and g^{-1} respectively, then $f \sim g$ iff $f^{-1} \sim g^{-1}$.

2 Natural transformations

2.1 The Yoneda lemma

A *natural transformation* between two functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of a family of morphisms $(\mu_C : FC \to GC)_{C \in \mathcal{C}_0}$ indexed by the collection of objects of $\mathcal{C}$, satisfying the following requirement: for every morphism $f : C \to C'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} FC \arrow[r, "\mu_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\mu_{C'}"'] & GC' \end{tikzcd}$$

commutes in $\mathcal{D}$ (the diagram above is called the naturality square). We say $\mu = (\mu_C)_{C \in \mathcal{C}_0} : F \Rightarrow G$ and we call μ_C the *component at* C of the natural transformation μ.

Given natural transformations $\mu : F \Rightarrow G$ and $\nu : G \Rightarrow H$ we have a natural transformation $\nu\mu = (\nu_C \mu_C)_C : F \Rightarrow H$, and with this composition there is a category $\mathcal{D}^{\mathcal{C}}$ with functors $F : \mathcal{C} \to \mathcal{D}$ as objects, and natural transformations as arrows.

One of the points of the naturality square condition in the definition of a natural transformation is given by the following proposition. Compare with the situation in Set: denoting the set of all functions from X to Y by Y^X, for any set Z there is a bijection between functions $Z \to Y^X$ and functions $Z \times X \to Y$ (Set is *cartesian closed*: see chapter 7).

Proposition 2.1 *For categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ there is a bijection:*

$$\mathrm{Cat}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Cat}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$$

Proof. Given $F : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$ define for every object E of $\mathcal{E}$ the functor $F_E : \mathcal{C} \to \mathcal{D}$ by $F_E(C) = F(E, C)$; for $f : C \to C'$ let $F_E(f) = F(\mathrm{id}_E, f) : F_E(C) = F(E, C) \to F(E, C') = F_E(C')$

Given $g : E \to E'$ in $\mathcal{E}$, the family $(F(g, \mathrm{id}_C) : F_E(C) \to F_{E'}(C))_{C \in \mathcal{C}_0}$ is a natural transformation: $F_E \Rightarrow F_{E'}$. So we have a functor $F \mapsto F_{(-)} : \mathcal{E} \to \mathcal{D}^{\mathcal{C}}$.

Conversely, given a functor $G : \mathcal{E} \to \mathcal{D}^{\mathcal{C}}$ we define a functor $\tilde{G} : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$ on objects by $\tilde{G}(E, C) = G(E)(C)$, and on arrows by $\tilde{G}(g, f) = G(g)_{C'} G(E)(f) = G(E')(f) G(g)_C$:

$$\begin{tikzcd} G(E)(C) = \tilde{G}(E, C) \arrow[r, "G(g)_C"] \arrow[d, "G(E)(f)"'] & \tilde{G}(E', C) = G(E')(C) \arrow[d, "G(E')(f)"] \\ \tilde{G}(E, C') \arrow[r, "G(g)_{C'}"'] & \tilde{G}(E', C') = G(E')(C') \end{tikzcd}$$

Exercise 18 Write out the details. Check that $\tilde{G}$ as just defined, is a functor, and that the two operations

$$\mathrm{Cat}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \rightleftarrows \mathrm{Cat}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$$

are inverse to each other.

∎

An important example of natural transformations arises from the functors $h_C : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (see example i) in the preceding chapter); defined on objects by $h_C(C') = \mathcal{C}(C', C)$ and on arrows $f : C'' \to C'$ so that $h_C(f)$ is composition with f: $\mathcal{C}(C', C) \to \mathcal{C}(C'', C)$.

Given $g : C_1 \to C_2$ there is a natural transformation

$$h_g : h_{C_1} \Rightarrow h_{C_2}$$

whose components are composition with g.

Exercise 19 Spell this out.

We have, in other words, a functor

$$h_{(-)} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

This functor is also often denoted by Y and answers to the name *Yoneda embedding.*

An embedding is a functor which is full and faithful and injective on objects. That Y is injective on objects is easy to see, because $\mathrm{id}_C \in h_C(C)$ for each object C, and id_C is in no other set $h_D(E)$; that Y is full and faithful follows from the following

Proposition 2.2 (Yoneda lemma) *For every object F of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and every object C of $\mathcal{C}$, there is a bijection $f_{C,F} : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C, F) \to F(C)$. Moreover, this bijection is* natural *in C and F in the following sense: given $g : C' \to C$ in $\mathcal{C}$ and $\mu : F \Rightarrow F'$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, the diagram*

$$\begin{tikzcd}
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C, F) \arrow[r, "f_{C,F}"] \arrow[d, "\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(g,\mu)"'] & F(C) \arrow[d, "\mu_{C'} F(g) = F'(g)\mu_C"] \\
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_{C'}, F') \arrow[r, "f_{C',F'}"'] & F'(C')
\end{tikzcd}$$

commutes in Set.

Proof. For every object C' of $\mathcal{C}$, every element f of $h_C(C') = \mathcal{C}(C', C)$ is equal to $\mathrm{id}_C f$ which is $h_C(f)(\mathrm{id}_C)$.

If $\kappa = (\kappa_{C'} | C' \in \mathcal{C}_0)$ is a natural transformation: $h_C \Rightarrow F$ then, $\kappa_{C'}(f)$ must be equal to $F(f)(\kappa_C(\mathrm{id}_C))$. So κ is completely determined by $\kappa_C(\mathrm{id}_C) \in F(C)$ and conversely, any element of $F(C)$ determines a natural transformation $h_C \Rightarrow F$.

Given $g : C' \to C$ in $\mathcal{C}$ and $\mu : F \Rightarrow F'$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, the map $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(g, \mu)$ sends the natural transformation $\kappa = (\kappa_{C''} | C'' \in \mathcal{C}_0) : h_C \Rightarrow F$ to $\lambda = (\lambda_{C''} | C'' \in \mathcal{C}_0)$ where $\lambda_{C''} : h_{C'}(C'') \to F'(C'')$ is defined by

$$\lambda_{C''}(h : C'' \to C') = \mu_{C''}(\kappa_{C''}(gh))$$

Now

$$\begin{array}{rcl} f_{C',F'}(\lambda) & = & \lambda_{C'}(\mathrm{id}_{C'}) \\ & = & \mu_{C'}(\kappa_{C'}(g)) \\ & = & \mu_{C'}(F(g)(\kappa_C(\mathrm{id}_C))) \\ & = & (\mu_{C'}F(g))(f_{C,F}(\kappa)) \end{array}$$

which proves the naturality statement. ∎

Corollary 2.3 *The functor $Y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is full and faithful.*

Proof. Immediate by the Yoneda lemma, since

$$\mathcal{C}(C, C') = h_{C'}(C) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C, h_{C'})$$

and this bijection is induced by Y. ∎

The use of the Yoneda lemma is often the following. One wants to prove that objects A and B of $\mathcal{C}$ are isomorphic. Suppose one can show that for every object X of $\mathcal{C}$ there is a bijection $f_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ which is natural in X; i.e. given $g : X' \to X$ in $\mathcal{C}$ one has that

$$\begin{tikzcd} \mathcal{C}(X,A) \arrow[r, "f_X"] \arrow[d, "\mathcal{C}(g,\mathrm{id}_A)"'] & \mathcal{C}(X,B) \arrow[d, "\mathcal{C}(g,\mathrm{id}_B)"] \\ \mathcal{C}(X',A) \arrow[r, "f_{X'}"'] & \mathcal{C}(X',B) \end{tikzcd}$$

commutes.

Then one can conclude that A and B are isomorphic in $\mathcal{C}$; for, from what one has just shown it follows that h_A and h_B are isomorphic objects in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$; that is, $Y(A)$ and $Y(B)$ are isomorphic. Since Y is full and faithful, A and B are isomorphic by the following exercise:

Exercise 20 Check: if $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, and $F(A)$ is isomorphic to $F(B)$ in $\mathcal{D}$, then A is isomorphic to B in $\mathcal{C}$.

Exercise 21 Suppose objects A and B are such that for every object X in $\mathcal{C}$ there is a bijection $f_X : \mathcal{C}(A, X) \to \mathcal{C}(B, X)$, naturally in a sense you define yourself. Conclude that A and B are isomorphic (hint: duality + the previous).

This argument can be carried further. Suppose one wants to show that two functors $F, G : \mathcal{C} \to \mathcal{D}$ are isomorphic as objects of $\mathcal{D}^{\mathcal{C}}$. Let's first spell out what this means:

Exercise 22 Show that F and G are isomorphic in $\mathcal{D}^{\mathcal{C}}$ if and only if there is a natural transformation $\mu : F \Rightarrow G$ such that all components μ_C are isomorphisms (in particular, if μ is such, the family $((\mu_C)^{-1} | C \in \mathcal{C}_0)$ is also a natural transformation $G \Rightarrow F$).

Now suppose one has for each $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$ a bijection

$$\mathcal{D}(D, FC) \cong \mathcal{D}(D, GC)$$

natural in D and C. This means that the objects h_{FC} and h_{GC} of $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ are isomorphic, by isomorphisms which are natural in C. By full and faithfulness of Y, FC and GC are isomorphic in $\mathcal{D}$ by isomorphisms natural in C; which says exactly that F and G are isomorphic as objects of $\mathcal{D}^{\mathcal{C}}$.

2.2 Examples of natural transformations

a) Let M and N be two monoids, regarded as categories with one object as in chapter 1. A functor $F : M \to N$ is then just the same as a homomorphism of monoids. Given two such, say $F, G : M \to N$, a natural transformation $F \Rightarrow G$ is (given by) an element n of N such that $nF(x) = G(x)n$ for all $x \in M$;

b) Let P and Q two preorders, regarded as categories. A functor $P \to Q$ is a monotone function, and there exists a unique natural transformation between two such, $F \Rightarrow G$, exactly if $F(x) \leq G(x)$ for all $x \in P$.

Exercise 23 In fact, show that if $\mathcal{D}$ is a preorder and the category $\mathcal{C}$ is *small*, i.e. the classes $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets, then the functor category $\mathcal{D}^{\mathcal{C}}$ is a preorder.

c) Let $U : \mathrm{Grp} \to \mathrm{Set}$ denote the forgetful functor, and $F : \mathrm{Set} \to \mathrm{Grp}$ the free functor (see chapter 1). There are natural transformations $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathrm{Grp}}$ and $\eta : \mathrm{id}_{\mathrm{Set}} \Rightarrow UF$.

   Given a group G, ε_G takes the string $\sigma = g_1 \ldots g_n$ to the product $g_1 \cdots g_n$ (here, the "formal inverses" g_i^{-1} are interpreted as the real inverses in G!).

   Given a set A, $\eta_A(a)$ is the singleton string a.

d) Let $i : \mathrm{Abgp} \to \mathrm{Grp}$ be the inclusion functor and $r : \mathrm{Grp} \to \mathrm{Abgp}$ the abelianization functor defined in example m) in chapter 1. There is $\varepsilon : ri \Rightarrow \mathrm{id}_{\mathrm{Abgp}}$ and $\eta : \mathrm{id}_{\mathrm{Grp}} \Rightarrow ir$.

   The components η_G of η are the quotient maps $G \to G/[G, G]$; the components of ε are isomorphisms.

e) There are at least two ways to associate a category to a set X: let $F(X)$ be the category with as objects the elements of X, and as only arrows identities (a category of the form $F(X)$ is called *discrete*; and $G(X)$ the category with the same objects but with exactly one arrow $f_{x,y} : x \to y$ for each pair (x, y) of elements of X (We might call $G(X)$ an *indiscrete category*).

Exercise 24 Check that F and G can be extended to functors: $\mathrm{Set} \to \mathrm{Cat}$ and describe the natural transformation $\mu : F \Rightarrow G$ which has, at each component, the identity function on objects.

f) Every class of arrows of a category $\mathcal{C}$ can be viewed as a natural transformation. Suppose $S \subseteq \mathcal{C}_1$. Let $F(S)$ be the discrete category on S as in the preceding example. There are the two functors $\mathrm{dom}, \mathrm{cod} : F(S) \to \mathcal{C}$, giving the domain and the codomain, respectively. For every $f \in S$ we have $f : \mathrm{dom}(f) \to \mathrm{cod}(f)$, and the family $(f | f \in S)$ defines a natural transformation: $\mathrm{dom} \Rightarrow \mathrm{cod}$.

g) Let A and B be sets. There are functors $(-) \times A : \mathrm{Set} \to \mathrm{Set}$ and $(-) \times B : \mathrm{Set} \to \mathrm{Set}$. Any function $f : A \to B$ gives a natural transformation $(-) \times A \Rightarrow (-) \times B$.

h) A category $\mathcal{C}$ is called a *groupoid* if every arrow of $\mathcal{C}$ is an isomorphism. Let $\mathcal{C}$ be a groupoid, and suppose we are given, for each object X of $\mathcal{C}$, an arrow μ_X in $\mathcal{C}$ with domain X.

Exercise 25 Show that there is a functor $F : \mathcal{C} \to \mathcal{C}$ in this case, which acts on objects by $F(X) = \mathrm{cod}(\mu_X)$, and that $\mu = (\mu_X | X \in \mathcal{C}_0)$ is a natural transformation: $\mathrm{id}_{\mathcal{C}} \Rightarrow F$.

i) Given categories $\mathcal{C}$, $\mathcal{D}$ and an object D of $\mathcal{D}$, there is the constant functor $\Delta_D : \mathcal{C} \to \mathcal{D}$ which assigns D to every object of $\mathcal{C}$ and id_D to every arrow of $\mathcal{C}$.

   Every arrow $f : D \to D'$ gives a natural transformation $\Delta_f : \Delta_D \Rightarrow \Delta_{D'}$ defined by $(\Delta_f)_C = f$ for each $C \in \mathcal{C}_0$.

j) Let $\mathcal{P}(X)$ denote the power set of a set X: the set of subsets of X. The powerset operation can be extended to a functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$. Given a function $f : X \to Y$ define $\mathcal{P}(f)$ by $\mathcal{P}(f)(A) = f[A]$, the image of $A \subseteq X$ under f.

   There is a natural transformation $\eta : \mathrm{id}_{\mathrm{Set}} \Rightarrow \mathcal{P}$ such that $\eta_X(x) = \{x\} \in \mathcal{P}(X)$ for each set X.

   There is also a natural transformation $\mu : \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}$. Given $A \in \mathcal{P}\mathcal{P}(X)$, so A is a set of subsets of X, we take its union $\bigcup(A)$ which is a subset of X. Put $\mu_X(A) = \bigcup(A)$.

2.3 Equivalence of categories; an example

As will become clear in the following chapters, equality between objects plays only a minor role in category theory. The most important categorical notions are only defined "up to isomorphism". This is in accordance with mathematical practice and with common sense: just renaming all elements of a group does not really give you another group.

We have already seen one example of this: the property of being a terminal object defines an object up to isomorphism. That is, any two terminal objects are isomorphic. There is, in the language of categories, no way of distinguishing between two isomorphic objects, so this is as far as we can get.

However, once we also consider functor categories, it turns out that there is another relation of "sameness" between categories, weaker than isomorphism of categories, and yet preserving all "good" categorical properties. Isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ requires the existence of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$; but bearing in mind that we can't really say meaningful things about equality between objects, we may relax the requirement by just asking that FG is *isomorphic* to $\mathrm{id}_{\mathcal{D}}$ in the functor category $\mathcal{D}^{\mathcal{D}}$ (and the same for GF); doing this we arrive at the notion of *equivalence of categories*, which is generally regarded as the proper notion of sameness.

So two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and natural transformations $\mu : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\nu : \mathrm{id}_{\mathcal{D}} \Rightarrow FG$ whose components are all isomorphisms. F and G are called *pseudo inverses* of each other. A functor which has a pseudo inverse is also called an *equivalence of categories.*

As a simple example of an equivalence of categories, take a preorder P. Let Q be the quotient of P by the equivalence relation which contains the pair (x, y) iff both $x \leq y$ and $y \leq x$ in P. Let $\pi : P \to Q$ be the quotient map. Regarding P and Q as categories, π is a functor, and in fact an equivalence of categories, though not in general an isomorphism.

Exercise 26 Work this out.

Exercise 27 Show that a category is equivalent to a discrete category if and only if it is a groupoid and a preorder.

In this section I want to give an important example of an equivalence of categories: the so-called "Lindenbaum-Tarski duality between Set and Complete Atomic Boolean Algebras". A duality between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence between $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}$ (equivalently, between $\mathcal{C}$ and $\mathcal{D}^{\mathrm{op}}$).

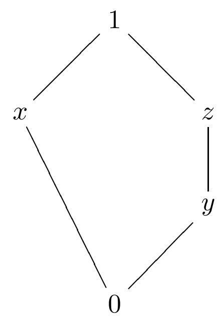
We need some definitions. A *lattice* is a partially ordered set in which every two elements x, y have a least upper bound (or join) $x \vee y$ and a greatest lower bound (or meet) $x \wedge y$; moreover, there exist a least element 0 and a greatest element 1.

Such a lattice is called a *Boolean algebra* if every element x has a *complement* $\neg x$, that is, satisfying $x \vee \neg x = 1$ and $x \wedge \neg x = 0$; and the lattice is *distributive*, which means that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z.

In a Boolean algebra, complements are unique, for if both y and z are complements of x, then

$$y = y \wedge 1 = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z$$

so $y \leq z$; similarly, $z \leq y$ so $y = z$. This is a non-example:



It is a lattice, and every element has a complement, but it is not distributive (check!).

A Boolean algebra B is *complete* if every subset A of B has a least upper bound $\bigvee A$ and a greatest lower bound $\bigwedge A$.

An *atom* in a Boolean algebra is an element x such that $0 < x$ but for no y we have $0 < y < x$. A Boolean algebra is atomic if every x is the join of the atoms below it:

$$x = \bigvee \{a | a \leq x \text{ and } a \text{ is an atom}\}$$

The category CABool is defined as follows: the objects are complete atomic Boolean algebras, and the arrows are complete homomorphisms, that is: $f : B \to C$ is a complete homomorphism if for every $A \subseteq B$,

$$f(\bigvee A) = \bigvee \{f(a) | a \in A\} \text{ and } f(\bigwedge A) = \bigwedge \{f(a) | a \in A\}$$

Exercise 28 Show that $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$. Conclude that a complete homomorphism preserves 1, 0 and complements.

Exercise 29 Show that $\bigwedge A = \neg \bigvee \{\neg a | a \in A\}$ and conclude that if a function preserves all $\bigvee$'s, 1 and complements, it is a complete homomorphism.

Theorem 2.4 *The category* CABool *is equivalent to* $\mathrm{Set}^{\mathrm{op}}$.

Proof. For every set X, $\mathcal{P}(X)$ is a complete atomic Boolean algebra and if $f : Y \to X$ is a function, then $f^{-1} : \mathcal{P}(X) \to \mathcal{P}(Y)$ which takes, for each subset of X, its inverse image under f, is a complete homomorphism. So this defines a functor $F : \mathrm{Set}^{\mathrm{op}} \to \mathrm{CABool}$.

Conversely, given a complete atomic Boolean algebra B, let $G(B)$ be the set of atoms of B. Given a complete homomorphism $g : B \to C$ we have a function $G(g) : G(C) \to G(B)$ defined by: $G(g)(c)$ is the unique $b \in G(B)$ such that $c \leq g(b)$. This is well-defined: first, there is an atom b with $c \leq g(b)$ because $1_B = \bigvee G(B)$ (B is atomic), so $1_C = g(1_B) = \bigvee\{g(b)|b \text{ is an atom}\}$ and:

Exercise 30 Prove: if c is an atom and $c \leq \bigvee A$, then there is $a \in A$ with $c \leq a$ (hint: prove for all a, b that $a \wedge b = 0 \Leftrightarrow a \leq \neg b$, and prove for a, c with c atom: $c \not\leq a \Leftrightarrow a \leq \neg c$).

Secondly, the atom b is unique since $c \leq g(b)$ and $c \leq g(b')$ means $c \leq g(b) \wedge g(b') = g(b \wedge b') = g(0) = 0$.

So we have a functor $G : \mathrm{CABool} \to \mathrm{Set}^{\mathrm{op}}$.

Now the atoms of the Boolean algebra $\mathcal{P}(X)$ are exactly the singleton subsets of X, so $GF(X) = \{\{x\}|x \in X\}$ which is clearly isomorphic to X. On the other hand, $FG(B) = \mathcal{P}(\{b \in B \,|\, b \text{ is an atom}\})$. There is a map from $FG(B)$ to B which sends each set of atoms to its least upper bound in B, and this map is an isomorphism in CABool. ∎

Exercise 31 Prove the last statement: that the map from $FG(B)$ to B, defined in the last paragraph of the proof, is an isomorphism.

Exercise 32 Prove that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is full and faithful, and *essentially surjective on objects*, that means: for any $D \in \mathcal{D}_0$ there is $C \in \mathcal{C}_0$ such that $F(C)$ is isomorphic to D.

3 (Co)cones and (Co)limits

3.1 Limits

Given a functor $F : \mathcal{C} \to \mathcal{D}$, a *cone* for F consists of an object D of $\mathcal{D}$ together with a natural transformation $\mu : \Delta_D \Rightarrow F$ (Δ_D is the constant functor with value D). In other words, we have a family $(\mu_C : D \to F(C) | C \in \mathcal{C}_0)$, and the naturality requirement in this case means that for every arrow $f : C \to C'$ in $\mathcal{C}$,

$$\begin{tikzcd} & D \arrow[dl, "\mu_C"'] \arrow[dr, "\mu_{C'}"] & \\ F(C) \arrow[rr, "F(f)"'] & & F(C') \end{tikzcd}$$

commutes in $\mathcal{D}$ (this diagram explains, I hope, the name "cone"). Let us denote the cone by (D, μ). D is called the *vertex* of the cone.

A *map of cones* $(D, \mu) \to (D', \mu')$ is a map $g : D \to D'$ such that $\mu'_C g = \mu_C$ for all $C \in \mathcal{C}_0$.

Clearly, there is a category $\mathrm{Cone}(F)$ which has as objects the cones for F and as morphisms maps of cones.

A *limiting cone* for F is a terminal object in $\mathrm{Cone}(F)$. Since terminal objects are unique up to isomorphism, as we have seen, any two limiting cones are isomorphic in $\mathrm{Cone}(F)$ and in particular, their vertices are isomorphic in $\mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is also called a *diagram* in $\mathcal{D}$ *of type* $\mathcal{C}$, and $\mathcal{C}$ is the *index category* of the diagram.

Let us see what it means to be a limiting cone, in some simple, important cases.

i) A limiting cone for the unique functor $! : \mathbf{0} \to \mathcal{D}$ ($\mathbf{0}$ is the empty category) "is" a terminal object in $\mathcal{D}$. For every object D of $\mathcal{D}$ determines, together with the empty family, a cone for !, and a map of cones is just an arrow in $\mathcal{D}$. So $\mathrm{Cone}(!)$ is isomorphic to $\mathcal{D}$.

ii) Let $\mathbf{2}$ be the discrete category with two objects x, y. A functor $\mathbf{2} \to \mathcal{D}$ is just a pair $\langle A, B \rangle$ of objects of $\mathcal{D}$, and a cone for this functor consists of an object C of $\mathcal{D}$ and two maps $\begin{tikzcd} C \arrow[r, "\mu_A"] \arrow[dr, "\mu_B"'] & A \\ & B \end{tikzcd}$ since there are no nontrivial arrows in $\mathbf{2}$.

$(C, (\mu_A, \mu_B))$ is a limiting cone for $\langle A, B \rangle$ iff the following holds: for any object D and arrows $f : D \to A$, $g : D \to B$, there is a unique arrow

$h : D \to C$ such that

$$\begin{tikzcd} & C \arrow[d, "\mu_A"] \arrow[ddr, "\mu_B"] & \\ D \arrow[ur, dashed, "h"] \arrow[r, "f"'] \arrow[drr, "g"'] & A & \\ & & B \end{tikzcd}$$

commutes. In other words, there is, for any D, a 1-1 correspondence between maps $D \to C$ and pairs of maps $\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$ This is the property of a *product*; a limiting cone for $\langle A, B\rangle$ is therefore called a product cone, and usually denoted:

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

The arrows π_A and π_B are called *projections.*

iii) Let $\hat{\mathbf{2}}$ denote the category $x \underset{b}{\overset{a}{\rightrightarrows}} y$. A functor $\hat{\mathbf{2}} \to \mathcal{D}$ is the same thing as a parallel pair of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in $\mathcal{D}$; I write $\langle f, g\rangle$ for this functor. A cone for $\langle f, g\rangle$ is:

$$\begin{tikzcd} & D \arrow[dl, "\mu_A"'] \arrow[dr, "\mu_B"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

But $\mu_B = f\mu_A = g\mu_A$ is already defined from μ_A, so giving a cone is the same as giving a map $\mu_A : D \to A$ such that $f\mu_A = g\mu_A$. Such a cone is limiting iff for any other map $h : C \to A$ with $fh = gh$, there is a unique $k : C \to D$ such that $h = \mu_A k$.

We call μ_A, if it is limiting, an *equalizer* of the pair f, g, and the diagram $D \xrightarrow{\mu_A} A \underset{g}{\overset{f}{\rightrightarrows}} B$ an equalizer diagram.

In Sets, an equalizer of f, g is isomorphic (as a cone) to the inclusion of $\{a \in A | f(a) = g(a)\}$ into A. In categorical interpretations of logical systems (see chapters 4 and 7), equalizers are used to interpret *equality between terms.*

Exercise 33 Show that every equalizer is a monomorphism.

Exercise 34 If $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an equalizer diagram, show that e is an isomorphism if and only if $f = g$.

Exercise 35 Show that in Set, every monomorphism fits into an equalizer diagram.

iv) Let J denote the category
$$\begin{tikzcd} & y \arrow[d, "b"] \\ x \arrow[r, "a"'] & z \end{tikzcd}$$
A functor $F : J \to \mathcal{D}$ is specified by giving two arrows in $\mathcal{D}$ with the same codomain, say $f : X \to Z$, $g : Y \to Z$. A limit for such a functor is given by an object W together with projections
$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \\ X & \end{tikzcd}$$
satisfying $fp_X = gp_Y$, and such that, given any other pair of arrows:
$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "s"'] & Y \\ X & \end{tikzcd}$$
with $gr = fs$, there is a unique arrow $V \to W$ such that
$$\begin{tikzcd} V \arrow[dr] \arrow[drr, "r"] \arrow[ddr, "s"'] & & \\ & W \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$
commutes.

The diagram
$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
is called a *pullback diagram.* In Set, the pullback cone for f, g is isomorphic to
$$\{(x, y) \in X \times Y | f(x) = g(y)\}$$
with the obvious projections.

We say that a category $\mathcal{D}$ *has binary products* (equalizers, pullbacks) iff every functor $\mathbf{2} \to \mathcal{D}$ ($\hat{\mathbf{2}} \to \mathcal{D}$, $J \to \mathcal{D}$, respectively) has a limiting cone. Some dependencies hold in this context:

Proposition 3.1 *If a category $\mathcal{D}$ has a terminal object and pullbacks, it has binary products and equalizers.*
If $\mathcal{D}$ has binary products and equalizers, it has pullbacks.

Proof. Let 1 be the terminal object in $\mathcal{D}$; given objects X and Y, if
$$\begin{tikzcd} C \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd}$$
is a pullback diagram, then
$$\begin{tikzcd} C \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \\ Y & \end{tikzcd}$$
is a product cone.

Given a product cone
$$\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \\ B & \end{tikzcd}$$
and maps
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & A \\ B & \end{tikzcd}$$
we write $X \stackrel{\langle f,g \rangle}{\to} A \times B$ for the unique factorization through the product. Write also $\delta : Y \to Y \times Y$ for $\langle \mathrm{id}_Y, \mathrm{id}_Y \rangle$.

Now given $f, g : X \to Y$, if

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d] & X \arrow[d, "\langle f,g \rangle"] \\ Y \arrow[r, "\delta"'] & Y \times Y \end{tikzcd}$$

is a pullback diagram, then $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an equalizer diagram. This proves the first statement.

As for the second: given
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
let $E \xrightarrow{e} X \times Y \overset{f\pi_X}{\underset{g\pi_Y}{\rightrightarrows}} Z$ be an equalizer; then

$$\begin{tikzcd} E \arrow[r, "\pi_X e"] \arrow[d, "\pi_Y e"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is a pullback diagram. ∎

Exercise 36 Let

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

a pullback diagram with f mono. Show that a is also mono. Also, if f is iso (an isomorphism), so is a.

Exercise 37 Given two commuting squares:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[r, "c"] \arrow[d, "f"'] & C \arrow[d, "d"'] \\ X \arrow[r, "g"'] & Y \arrow[r, "h"'] & Z \end{tikzcd}$$

a) if both squares are pullback squares, then so is the composite square
$$\begin{tikzcd} A \arrow[r, "cb"] \arrow[d, "a"'] & C \arrow[d, "d"] \\ X \arrow[r, "hg"'] & Z \end{tikzcd}$$

b) If the right hand square and the composite square are pullbacks, then so is the left hand square.

Exercise 38 $f : A \to B$ is a monomorphism if and only if

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback diagram.

A monomorphism $f : A \to B$ which fits into an equalizer diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

is called a *regular mono.*

Exercise 39 If

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & X \arrow[d, "g"] \\ B \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback and g is regular mono, so is b.

Exercise 40 If f is regular mono and epi, f is iso. Every split mono is regular.

Exercise 41 Give an example of a category in which not every mono is regular.

Exercise 42 In Grp, every mono is regular [This is not so easy].

Exercise 43 Characterize the regular monos in Pos.

Exercise 44 If a category $\mathcal{D}$ has binary products and a terminal object, it has *all finite products*, i.e. limiting cones for every functor into $\mathcal{D}$ from a finite discrete category.

Exercise 45 Suppose $\mathcal{C}$ has binary products and suppose for every ordered pair (A, B) of objects of $\mathcal{C}$ a product cone $\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \\ B & \end{tikzcd}$ has been chosen.

a) Show that there is a functor: $\mathcal{C} \times \mathcal{C} \xrightarrow{- \times -} \mathcal{C}$ (the product functor) which sends each pair (A, B) of objects to $A \times B$ and each pair of arrows $(f : A \to A', g : B \to B')$ to $f \times g = \langle f\pi_A, g\pi_B \rangle$.

b) From a), there are functors:

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \underset{- \times (- \times -)}{\overset{(- \times -) \times -}{\rightrightarrows}} \mathcal{C}$$

sending (A, B, C) to $\begin{matrix} (A \times B) \times C \\ A \times (B \times C) \end{matrix}$ Show that there is a natural transformation $a = (a_{A,B,C} | A, B, C \in \mathcal{C}_0)$ from $(- \times -) \times -$ to $- \times (- \times -)$

such that for any four objects A, B, C, D of $\mathcal{C}$:

$$\begin{tikzcd}
((A \times B) \times C) \times D \arrow[rr, "a_{A\times B,C,D}"] \arrow[d, "a_{A,B,C}\times \mathrm{id}_D"'] & & (A \times B) \times (C \times D) \arrow[d, "a_{A,B,C\times D}"] \\
(A \times (B \times C)) \times D \arrow[dr, "a_{A,B\times C,D}"'] & & A \times (B \times (C \times D)) \\
& A \times ((B \times C) \times D) \arrow[ur, "\mathrm{id}_A \times a_{B,C,D}"'] &
\end{tikzcd}$$

commutes (This diagram is called "MacLane's pentagon").

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits of type* $\mathcal{E}$ if for all functors $M : \mathcal{E} \to \mathcal{C}$, if (D, μ) is a limiting cone for M in $\mathcal{C}$, then $(FD, F\mu = (F(\mu_E)|E \in \mathcal{E}_0))$ is a limiting cone for FM in $\mathcal{D}$.

So, a functor $F : \mathcal{C} \to \mathcal{D}$ preserves binary products if for every product diagram $\begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \\ A & \end{tikzcd}$ its F-image $\begin{tikzcd} F(A \times B) \arrow[r, "F(\pi_B)"] \arrow[d, "F(\pi_A)"'] & F(B) \\ F(A) & \end{tikzcd}$ is again a product diagram. Similarly for equalizers and pullbacks.

Some more terminology: F is said to *preserve all finite limits* if it preserves limits of type $\mathcal{E}$ for every finite $\mathcal{E}$. A category which has all finite limits is called *lex* (*left exact*), *cartesian* or *finitely complete*.

Exercise 46 If a category $\mathcal{C}$ has equalizers, it has *all finite equalizers*: for every category $\mathcal{E}$ of the form

$$X \underset{f_n}{\overset{f_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\vdots}}}} Y$$

every functor $\mathcal{E} \to \mathcal{C}$ has a limiting cone.

Exercise 47 Suppose $F : \mathcal{C} \to \mathcal{D}$ preserves equalizers (and $\mathcal{C}$ has equalizers) and reflects isomorphisms. Then F is faithful.

Exercise 48 Let $\mathcal{C}$ be a category with finite limits. Show that for every object C of $\mathcal{C}$, the slice category $\mathcal{C}/C$ (example j) of 1.1) has binary products: the vertex of a product diagram for two objects $D \to C$, $D' \to C$ is $D'' \to C$ where

$$\begin{tikzcd}
D'' \arrow[r] \arrow[d] & D \arrow[d] \\
D' \arrow[r] & C
\end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

3.2 Limits by products and equalizers

In Set, every small diagram has a limit; given a functor $F : \mathcal{E} \to \text{Set}$ with $\mathcal{E}$ small, there is a limiting cone for F in Set with vertex

$$\{(x_E)_{E\in\mathcal{E}_0} \in \prod_{E\in\mathcal{E}_0} F(E)\,|\,\forall E \xrightarrow{f} E' \in \mathcal{E}_1(F(f)(x_E) = x_{E'})\}$$

So in Set, limits are equationally defined subsets of suitable products. This holds in any category:

Proposition 3.2 *Suppose $\mathcal{C}$ has all small products (including the empty product, i.e. a terminal object 1) and equalizers; then $\mathcal{C}$ has all small limits.*

Proof. Given a set I and an I-indexed family of objects $(A_i|i \in I)$ of $\mathcal{C}$, we denote the product by $\prod_{i\in I} A_i$ and projections by $\pi_i : \prod_{i\in I} A_i \to A_i$; an arrow $f : X \to \prod_{i\in I} A_i$ which is determined by the compositions $f_i = \pi_i f : X \to A_i$, is also denoted $(f_i|i \in I)$.

Now given $\mathcal{E} \to \mathcal{C}$ with $\mathcal{E}_0$ and $\mathcal{E}_1$ sets, we construct

$$E \xrightarrow{e} \prod_{i\in\mathcal{E}_0} F(i) \underset{(F(u)\pi_{\text{dom}(u)}|u\in\mathcal{E}_1)}{\overset{(\pi_{\text{cod}(u)}|u\in\mathcal{E}_1)}{\rightrightarrows}} \prod_{u\in\mathcal{E}_1} F(\text{cod}(u))$$

in $\mathcal{C}$ as an equalizer diagram. The family $(\mu_i = \pi_i e : E \to F(i)|i \in \mathcal{E}_0)$ is a natural transformation $\Delta_E \Rightarrow F$ because, given an arrow $u \in \mathcal{E}_1$, say $u : i \to j$, we have that

$$\begin{tikzcd} & E \arrow[dl, "\pi_i e"'] \arrow[dr, "\pi_j e"] & \\ F(i) \arrow[rr, "F(u)"'] & & F(j) \end{tikzcd}$$

commutes since $F(u)\pi_i e = F(u)\pi_{\text{dom}(u)}e = \pi_{\text{cod}(u)}e = \pi_j e$.

So (E, μ) is a cone for F, but every other cone (D, ν) for F gives a map $d : D \to \prod_{i\in\mathcal{E}_0} F(i)$ equalizing the two horizontal arrows; so factors uniquely through E. ∎

Exercise 49 Check that "small" in the statement of the proposition, can be replaced by "finite": if $\mathcal{C}$ has all finite products and equalizers, $\mathcal{C}$ is finitely complete.

Exercise 50 Show that if $\mathcal{C}$ is complete, then $F : \mathcal{C} \to \mathcal{D}$ preserves all limits if F preserves products and equalizers. This no longer holds if $\mathcal{C}$ is not complete! That is, F may preserve all products and equalizers which exist in $\mathcal{C}$, yet not preserve all limits which exist in $\mathcal{C}$.

3.3 Complete Categories

A category is called *complete* if it has limits of type $\mathcal{E}$ for all small $\mathcal{E}$.

In general, limits over large (i.e. not small) diagrams need not exist. For example in Set, there is a limiting cone for the identity functor Set $\to$ Set (its vertex is the empty set), but not for the constant functor $\Delta_A : \mathcal{C} \to \mathrm{Set}$ if $\mathcal{C}$ is a large discrete category and A has more than one element.

The categories Set, Top, Pos, Mon, Grp, Grph, Rng,... are all complete. For instance in Top, the product of a set $(X_i \mid i \in I)$ of topological spaces is the set $\prod_{i\in I} X_i$ with the product topology; the equalizer of two continuous maps $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is the inclusion $X' \subseteq X$ where $X' = \{x \in X \mid f(x) = g(x)\}$ with the subspace topology from X.

Exercise 51 What are monomorphisms in Top? Is every mono regular in Top?

The category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is also complete, and limits are "computed pointwise". That is, let $F : \mathcal{D} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ be a diagram in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. For every $C \in \mathcal{C}_0$ there is a functor $F_C : \mathcal{D} \to \mathrm{Set}$, given by $F_C(D) = F(D)(C)$ and for $f : D \to D'$ in $\mathcal{D}$, $F_C(f) = F(f)_C : F(D)(C) \to F(D')(C)$.

Since Set is complete, every F_C has a limiting cone (X_C, μ_C) in Set. Now if $C' \xrightarrow{g} C$ is a morphism in $\mathcal{C}$, the collection of arrows

$$\{X_C \overset{(\mu_C)_D}{\to} F(D)(C) \overset{F(D)(g)}{\to} F(D)(C') = F_{C'}(D) \mid D \in \mathcal{D}_0\}$$

is a cone for $F_{C'}$ with vertex X_C, since for any $f : D \to D'$ we have $F(f)_{C'} \circ F(D)(g) \circ (\mu_C)_D = F(D')(g) \circ F(f)_C \circ (\mu_C)_D$ (by naturality of $F(f)$) $= F(D')(g) \circ (\mu_C)_{D'}$ (because (X_C, μ_C) is a cone).

Because $(X_{C'}, \mu_{C'})$ is a limiting cone for $F_{C'}$, there is a unique arrow $X_g : X_C \to X_{C'}$ in Set such that $F(D)(g) \circ (\mu_C)_D = (\mu_{C'})_D \circ X_g$ for all $D \in \mathcal{D}_0$. By the uniqueness of these arrows, we have an object X of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, and arrows $\nu_D : X \to F(D)$ for all $D \in \mathcal{D}_0$, and the pair (X, ν) is a limiting cone for F in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Exercise 52 Check the remaining details.

It is a consequence of the Yoneda lemma that the Yoneda embedding $Y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. For, let $F : \mathcal{D} \to \mathcal{C}$ be a diagram with limiting cone (E, ν) and let (X, δ) be a limiting cone for the composition $Y \circ F : \mathcal{D} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. By the Yoneda lemma, $X(C)$ is in natural 1-1 correspondence with the set of arrows $Y(C) \to X$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$; which by the fact that (X, δ) is a limiting cone, is in natural 1-1 correspondence with cones $(Y(C), \mu)$ for $Y \circ F$ with vertex $Y(C)$; since Y is full and faithful every such cone comes from a unique cone (C, μ') for F in $\mathcal{C}$, hence from a unique map $C \to E$ in $\mathcal{C}$.

So, $X(C)$ is naturally isomorphic to $\mathcal{C}(C, E)$ whence X is isomorphic to $Y(E)$, by an isomorphism which transforms δ into $Y \circ \nu = (Y(\nu_C) \mid C \in \mathcal{C}_0)$.

To finish this section a little theorem by Peter Freyd which says that every small, complete category is a complete preorder:

Proposition 3.3 *Suppose $\mathcal{C}$ is small and complete. Then $\mathcal{C}$ is a preorder.*

Proof. If not, there are objects A, B in $\mathcal{C}$ such that there are two distinct maps $f, g : A \to B$. Since $\mathcal{C}_1$ is a set and $\mathcal{C}$ complete, the product $\prod_{h\in\mathcal{C}_1} B$ exists. Arrows $k : A \to \prod_{h\in\mathcal{C}_1} B$ are in 1-1 correspondence with families of arrows $(k_h : A \to B \mid h \in \mathcal{C}_1)$. For every subset $X \subseteq \mathcal{C}_1$ define such a family by:

$$k_h = \begin{cases} f & \text{if } h \in X \\ g & \text{else} \end{cases}$$

This gives an injective function from $2^{\mathcal{C}_1}$ into $\mathcal{C}(A, \prod_{h\in\mathcal{C}_1} B)$ hence into $\mathcal{C}_1$, contradicting Cantor's theorem in set theory. ∎

3.4 Colimits

The dual notion of limit is colimit. Given a functor $F : \mathcal{E} \to \mathcal{C}$ there is clearly a functor $F^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ which does "the same" as F. We say that a *colimiting cocone* for F is a limiting cone for F^{op}.

So: a cocone for $F : \mathcal{E} \to \mathcal{C}$ is a pair (ν, D) where $\nu : F \Rightarrow \Delta_D$ and a colimiting cocone is an initial object in the category Cocone(F).

Examples

i) a colimiting cocone for $! : \mathbf{0} \to \mathcal{C}$ "is" an initial object of $\mathcal{C}$

ii) a colimiting cocone for $\langle A, B\rangle : \mathbf{2} \to \mathcal{C}$ is a *coproduct* of A and B in $\mathcal{C}$: usually denoted $A + B$ or $A \sqcup B$; there are *coprojections* or *coproduct inclusions*

$$\begin{tikzcd} A \arrow[dr, "\nu_A"] & \\ B \arrow[r, "\nu_B"'] & A \sqcup B \end{tikzcd}$$

with the property that, given any pair of arrows $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ there is a unique map $\begin{bmatrix} f \\ g \end{bmatrix} : A \sqcup B \to C$ such that $f = \begin{bmatrix} f \\ g \end{bmatrix} \nu_A$ and $g = \begin{bmatrix} f \\ g \end{bmatrix} \nu_B$

iii) a colimiting cocone for $A \overset{f}{\underset{g}{\rightrightarrows}} B$ (as functor $\hat{\mathbf{2}} \to \mathcal{C}$) is given by a map $B \xrightarrow{c} C$ satisfying $cf = cg$, and such that for any $B \xrightarrow{h} D$ with $hf = hg$ there is a unique $C \xrightarrow{h'} D$ with $h = h'c$. c is called a *coequalizer* for f and g; the diagram $A \rightrightarrows B \longrightarrow C$ a coequalizer diagram.

Exercise 53 Is the terminology "coproduct inclusions" correct? That is, it suggests they are monos. Is this always the case?

Formulate a condition on A and B which implies that ν_A and ν_B are monic.

In Set, the coproduct of X and Y is the disjoint union $(\{0\} \times X) \cup (\{1\} \times Y)$ of X and Y. The coequalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the quotient map $Y \to Y/\sim$ where $\sim$ is the equivalence relation generated by

$$y \sim y' \text{ iff there is } x \in X \text{ with } f(x) = y \text{ and } g(x) = y'$$

The dual notion of pullback is *pushout.* A pushout diagram is a colimiting cocone for a functor $\Gamma \to \mathcal{C}$ where Γ is the category

$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd}$$

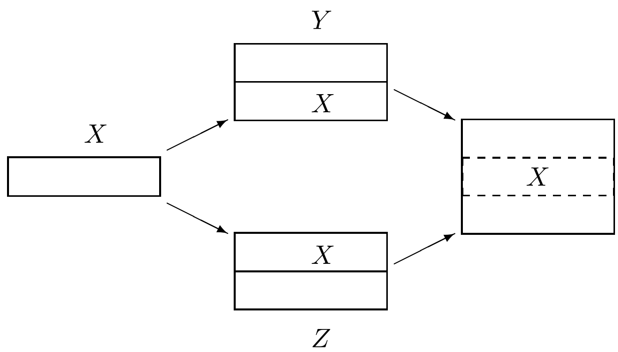
. Such a diagram is a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "a"] \\ Z \arrow[r, "b"'] & P \end{tikzcd}$$

which commutes and such that, given

$$\begin{tikzcd} Y \arrow[dr, "\alpha"] & \\ Z \arrow[r, "\beta"'] & Q \end{tikzcd}$$

with $\alpha f = \beta g$, there is a unique $P \xrightarrow{p} Q$ with $\alpha = pa$ and $\beta = pb$. In Set, the pushout of $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ is the coproduct $Y \sqcup Z$ where the two images of X are identified:



Exercise 54 Give yourself, in terms of $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$, a formal definition of a relation R on $Y \sqcup Z$ such that the pushout of f and g is $Y \sqcup Z/\sim$, $\sim$ being the equivalence relation generated by R.

One can now dualize every result and exercise from the section on limits:

Exercise 55 f is epi if and only if

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "f"'] & {} \arrow[d, "\mathrm{id}"] \\ {} \arrow[r, "\mathrm{id}"'] & {} \end{tikzcd}$$

is a pushout diagram.

Exercise 56 Every coequalizer is an epimorphism; if e is a coequalizer of f and g, then e is iso iff $f = g$

Exercise 57 If $\mathcal{C}$ has an initial object and pushouts, $\mathcal{C}$ has binary coproducts and coequalizers; if $\mathcal{C}$ has binary coproducts and coequalizers, $\mathcal{C}$ has pushouts.

Exercise 58 If $\begin{tikzcd} {} \arrow[r] \arrow[d, "a"'] & {} \arrow[d, "f"] \\ {} \arrow[r] & {} \end{tikzcd}$ is a pushout diagram, then a epi implies f epi, and a regular epi (i.e. a coequalizer) implies f regular epi.

Exercise 59 The composition of two puhout squares is a pushout; if both the first square and the composition are pushouts, the second square is.

Exercise 60 If $\mathcal{C}$ has all small (finite) coproducts and coequalizers, $\mathcal{C}$ has all small (finite) colimits.

Some miscellaneous exercises:

Exercise 61 Call an arrow f a *stably regular epi* if whenever $\begin{tikzcd} {} \arrow[r] \arrow[d, "a"'] & {} \arrow[d, "f"] \\ {} \arrow[r] & {} \end{tikzcd}$ is a pullback diagram, the arrow a is a regular epi. Show: in Pos, $X \xrightarrow{f} Y$ is a stably regular epi if and only if for all y, y' in Y:

$$y \leq y' \Leftrightarrow \exists x \in f^{-1}(y) \exists x' \in f^{-1}(y').x \leq x'$$

Show by an example that not every epi is stably regular in Pos.

Exercise 62 In Grp, every epi is regular.

Exercise 63 Characterize coproducts in Abgrp.

4 A little piece of categorical logic

One of the major achievements of category theory in mathematical logic and in computer science, has been a unified treatment of semantics for all kinds of logical systems and term calculi which are the basis for programming languages.

One can say that mathematical logic, seen as the study of classical first order logic, first started to be a real subject with the discovery, by Gödel, of the *completeness theorem* for set-theoretic interpretations: a sentence φ is provable if and only if φ is true in all possible interpretations. This unites the two approaches to logic: proof theory and model theory, makes logic accessible for mathematical methods and enables one to give nice and elegant proofs of proof theoretical properties by model theory (for example, the Beth and Craig definability and interpolation theorems).

However the completeness theorem needs generalization once one considers logics, such as intuitionistic logic (which does not admit the principle of excluded middle), minimal logic (which has no negation) or modal logic (where the logic has an extra operator, expressing "necessarily true"), for which the set-theoretic interpretation is not complete. One therefore comes with a general definition of "interpretation" in a category $\mathcal{C}$ of a logical system, which generalizes Tarski's truth definition: this will then be the special case of classical logic and the category Set.

In this chapter I treat, for reasons of space, only a fragment of first order logic: regular logic. On this fragment the valid statements of classical and intuitionistic logic coincide.

For an interpretation of a term calculus like the λ-calculus, which is of paramount importance in theoretical computer science, the reader is referred to chapter 7.

4.1 Regular categories and subobjects

Definition 4.1 *A category $\mathcal{C}$ is called* regular *if the following conditions hold:*

a) *$\mathcal{C}$ has all finite limits;*

b) *For every arrow f, if*

$$\begin{tikzcd} Z \arrow[r, "p_0"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback (then $Z \overset{p_0}{\underset{p_1}{\rightrightarrows}} X$ is called the kernel pair *of f), the coequalizer of p_0, p_1 exists;*

c) *Regular epimorphisms (coequalizers) are* stable under pullback*, that is: in* *a pullback square*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "a"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

if f is regular epi, so is a.

Examples. In Set, as in Grp, Top, etc., the (underlying) set which is the vertex of the kernel pair of $f : X \to Y$ is $X_f = \{(x, x') | \, f(x) = f(x')\}$. The coequalizer of $X_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$ is (up to isomorphism) the map $X \to \mathrm{Im}(f)$ where $\mathrm{Im}(f)$ is the set-theoretic image of f as subset of Y.

These coequalizers exist in Set, Top, Grp, Pos.... Moreover, in Set and Grp every epi is regular, and (since epis in Set and Grp are just surjective functions) stable under pullback; hence Set and Grp are examples of regular categories.

Top is not regular! It satisfies the first two requirements of the definition, but not the third. One can prove that the functor $(-)\times S : \mathrm{Top} \to \mathrm{Top}$ preserves all quotient maps only if the space S is locally compact. Since every coequalizer is a quotient map, if S is not locally compact there will be pullbacks of form

$$\begin{tikzcd} X \times S \arrow[r, "f \times \mathrm{id}_S"] \arrow[d] & Y \times S \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with f regular epi, but $f \times \mathrm{id}_S$ not.

Exercise 64 This exercise shows that Pos is not regular either. Let X and Y be the partial orders $\{x \leq y, y' \leq z\}$ and $\{a \leq b \leq c\}$ respectively.

a) Prove that $f(x) = a$, $f(y) = f(y') = b$, $f(z) = c$ defines a regular epimorphism: $X \to Y$.

b) Let Z be $\{a \leq c\} \subset Y$ and $W = f^{-1}(Z) \subset X$. Then

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback, but $W \to Z$ is not the coequalizer of anything.

Proposition 4.2 *In a regular category, every arrow $f : X \to Y$ can be factored as $f = me : X \xrightarrow{e} E \xrightarrow{m} Y$ where e is regular epi and m is mono; and this factorization is unique in the sense that if f is also $m'e' : X \xrightarrow{e'} E' \xrightarrow{m'} Y$ with m' mono and e' regular epi, there is an isomorphism $\sigma : E \to E'$ such that $\sigma e = e'$ and $m'\sigma = m$.*

Proof. Given $f : X \to Y$ we let $X \xrightarrow{e} E$ be the coequalizer of the kernel pair $Z \overset{p_0}{\underset{p_1}{\rightrightarrows}} X$ of f. Since $fp_0 = fp_1$ there is a unique $m : E \to Y$ such that $f = me$. By construction e is regular epi; we must show that m is mono, and the uniqueness of the factorization.

Suppose $mg = mh$ for $g, h : W \to E$; we prove that $g = h$. Let

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d, "\langle q_0, q_1 \rangle"'] & W \arrow[d, "\langle g, h \rangle"] \\ X \times X \arrow[r, "e \times e"'] & E \times E \end{tikzcd}$$

be a pullback square. Then

$$fq_0 = meq_0 = mga = mha = meq_1 = fq_1$$

so there is a unique arrow $V \xrightarrow{b} Z$ such that $\langle q_0, q_1 \rangle = \langle p_0, p_1 \rangle b : V \to X \times X$ (because of the kernel pair property). It follows that

$$ga = eq_0 = ep_0 b = ep_1 b = eq_1 = ha$$

I claim that a is epi, so it follows that $g = h$. It is here that we use the requirement that regular epis are stable under pullback. Now $e \times e : X \times X \to E \times E$ is the composite

$$X \times X \overset{e \times \mathrm{id}_X}{\to} E \times X \overset{\mathrm{id}_E \times e}{\to} E \times E$$

and both maps are regular epis since both squares

$$\begin{tikzcd} X \times X \arrow[r, "e \times \mathrm{id}_X"] \arrow[d, "\pi_0"'] & E \times X \arrow[d, "\pi_0"] \\ X \arrow[r, "e"'] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \times X \arrow[r, "\mathrm{id}_E \times e"] \arrow[d, "\pi_1"'] & E \times E \arrow[d, "\pi_1"] \\ X \arrow[r, "e"'] & E \end{tikzcd}$$

are pullbacks. The map a, being the pullback of a composite of regular epis, is then itself the composite of regular epis (check this!), so in particular epi.

This proves that m is mono, and we have our factorization.

As to uniqueness, suppose we had another factorization $f = m'e'$ with m' mono and e' regular epi. Then $m'e'p_0 = fp_0 = fp_1 = m'e'p_1$ so since m' mono, $e'p_0 = e'p_1$. Because e is the coequalizer of p_0 and p_1, there is a unique σ:

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[dr, "e'"'] & {} \arrow[d, "\sigma"] \\ & {} \end{tikzcd}$$

Then $m'\sigma e = m'e' = f = me$ so since e epi, $m = m'\sigma$.

Now $e' : X \to E'$ is a coequalizer; say $U \overset{k}{\underset{l}{\rightrightarrows}} X \xrightarrow{e'} E'$ is a coequalizer diagram. Then it follows that $ek = el$ (since $mek = m'e'k = m'e'l = mel$

and m mono) so there is a unique τ: $\begin{tikzcd} \arrow[r, "e"] \arrow[dr, "e'"'] & {} \\ & {} \arrow[u, "\tau"'] \end{tikzcd}$ Then $m\tau\sigma e = m\tau e' = me$; since m mono and e epi, $\tau\sigma = \mathrm{id}_E$. Similarly, $\sigma\tau = \mathrm{id}_{E'}$. So σ is the required isomorphism. ∎

Exercise 65 Check this detail: in a regular category $\mathcal{C}$, if is a pullback diagram and $b = c_1c_2$ with c_1 and c_2 regular epis, then $a = a_1a_2$ for some regular epis a_1, a_2.

Subobjects. In any category $\mathcal{C}$ we define a *subobject* of an object X to be an equivalence class of monomorphisms $Y \xrightarrow{m} X$, where $Y \xrightarrow{m} X$ is equivalent to $Y' \xrightarrow{m'} X$ if there is an isomorphism $\sigma : Y \to Y'$ with $m'\sigma = m$ (then $m\sigma^{-1} = m'$ follows). We say that $Y \xrightarrow{m} X$ represents a *smaller* subobject than $Y' \xrightarrow{m'} X$ if there is $\sigma : Y \to Y'$ with $m'\sigma = m$ (σ not necessarily iso; but check that σ is always mono).

The notion of subobject is the categorical version of the notion of subset in set theory. In Set, two injective functions represent the same subobject iff their images are the same; one can therefore identify subobjects with subsets. Note however, that in Set we have a "canonical" *choice* of representative for each subobject: the inclusion of the subset to which the subobject corresponds. This choice is not always available in general categories.

We have a partial order Sub(X) of subobjects of X, ordered by the smaller-than relation.

Proposition 4.3 *In a category $\mathcal{C}$ with finite limits, each pair of elements of Sub(X) has a greatest lower bound. Moreover, Sub(X) has a largest element.*

Proof. If $Y \xrightarrow{m} X$ and $Y' \xrightarrow{m'} X$ represent two subobjects of X and
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "m"] \\ Y' \arrow[r, "m'"'] & X \end{tikzcd}$$
is a pullback, then $Z \to X$ is mono, and represents the greatest lower bound (check!).

Of course, the identity $X \xrightarrow{\mathrm{id}_X} X$ represents the top element of Sub(X). ∎

Because the factorization of $X \xrightarrow{f} Y$ as $X \xrightarrow{e} E \xrightarrow{m} Y$ with e regular epi and m mono, in a regular category $\mathcal{C}$, is only defined up to isomorphism, it defines rather a subobject of Y than a mono into Y; this defined subobject is called the *image* of f and denoted Im(f) (compare with the situation in Set).

Exercise 66 Im(f) is the smallest subobject of Y through which f factors: for a subobject represented by $n : A \to Y$ we have that there is $X \xrightarrow{a} A$ with $f = na$, if and only if Im(f) is smaller than the subobject represented by n.

Since monos and isos are stable under pullback, in any category $\mathcal{C}$ with pullbacks, any arrow $f : X \to Y$ determines an order preserving map $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ *by pullback along* f: if $E \stackrel{m}{\to} Y$ represents the subobject M of Y and
$$\begin{tikzcd} F \arrow[r] \arrow[d, "n"'] & E \arrow[d, "m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
is a pullback, $F \stackrel{n}{\to} X$ represents $f^*(M)$.

Exercise 67 Check that f^* is well defined and order preserving.

Proposition 4.4 *In a regular category, each f^* preserves greatest lower bounds and images, that is: for $f : X \to Y$,*

i) *for subobjects M, N of Y, $f^*(M \wedge N) = f^*(M) \wedge f^*(N)$;*

ii) *if*
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "g'"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$
is a pullback, then $f^*(\mathrm{Im}(g)) = \mathrm{Im}(g')$.

Exercise 68 Prove proposition 4.4.

Exercise 69 Suppose $f : X \to Y$ is an arrow in a regular category. For a subobject M of X, represented by a mono $E \stackrel{m}{\to} X$, write $\exists_f(M)$ for the subobject $\mathrm{Im}(fm)$ of Y.

a) Show that $\exists_f(M)$ is well-defined, that is: depends only on M, not on the representative m.

b) Show that if $M \in \mathrm{Sub}(X)$ and $N \in \mathrm{Sub}(Y)$, then $\exists_f(M) \leq N$ if and only if $M \leq f^*(N)$.

4.2 The logic of regular categories

The fragment of first order logic we are going to interpret in regular categories is the so-called regular fragment.

The logical symbols are $=$ (equality), $\wedge$ (conjunction) and $\exists$ (existential quantification). A *language* consists of a set of *sorts* $S, T, \ldots$; a denumerable collection of *variables* $x_1^S, x_2^S, \ldots$ of sort S, for each sort; a collection of *function symbols* $(f : S_1, \ldots, S_n \to S)$ and *relation symbols* $(R \subseteq S_1, \ldots, S_m)$. The case $n = 0$ is not excluded (one thinks of *constants* of a sort in case of 0-placed function symbols, and of *atomic propositions* in the case of 0-placed relation symbols), but not separately treated. We now define, inductively, terms of sort S and formulas.

Definition 4.5 *Terms of sort S are defined by:*

i) *x^S is a term of sort S if x^S is a variable of sort S;*

ii) *if* $t_1, \ldots, t_n$ *are terms of sorts* $S_1, \ldots, S_n$ *respectively, and*

$$(f : S_1, \ldots S_n \to S)$$

is a function symbol of the language, then $f(t_1, \ldots, t_n)$ *is a term of sort* S*.*

Formulas are defined by:

i) $\top$ *is a formula (the formula "true");*

ii) *if* t *and* s *are terms of the same sort, then* $t = s$ *is a formula;*

iii) *if* $(R \subseteq S_1, \ldots, S_m)$ *is a relation symbol and* $t_1, \ldots, t_m$ *are terms of sorts* $S_1, \ldots, S_m$ *respectively, then* $R(t_1, \ldots, t_m)$ *is a formula;*

iv) *if* φ *and* ψ *are formulas then* $(\varphi \wedge \psi)$ *is a formula;*

v) *if* φ *is a formula and* x *a variable of some sort, then* $\exists x \varphi$ *is a formula.*

An *interpretation* of such a language in a regular category $\mathcal{C}$ is given by choosing for each sort S an object $[\![S]\!]$ of $\mathcal{C}$, for each function symbol $(f : S_1, \ldots, S_n \to S)$ of the language, an arrow $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![S]\!]$ in $\mathcal{C}$, and for each relation symbol $(R \subseteq S_1, \ldots, S_m)$ a subobject $[\![R]\!]$ of $[\![S_1]\!] \times \cdots \times [\![S_m]\!]$.

Given this, we define interpretations $[\![t]\!]$ for terms t and $[\![\varphi]\!]$ for formulas φ, as follows.

Write $FV(t)$ for the set of variables which occur in t, and $FV(\varphi)$ for the set of *free* variables in φ.

We put $[\![FV(t)]\!] = [\![S_1]\!] \times \cdots \times [\![S_n]\!]$ if $FV(t) = \{x_1^{S_1}, \ldots, x_n^{S_n}\}$; the same for $[\![FV(\varphi)]\!]$. Note: in the products $[\![FV(t)]\!]$ and $[\![FV(\varphi)]\!]$ we take a copy of $[\![S]\!]$ for every variable of sort S! Let me further emphasize that the empty product is 1, so if $FV(t)$ (or $FV(\varphi)$) is $\emptyset$, $[\![FV(t)]\!]$ (or $[\![FV(\varphi)]\!]$) is the terminal object of the category.

Definition 4.6 *The interpretation of a term* t *of sort* S *is a morphism* $[\![t]\!] : [\![FV(t)]\!] \to [\![S]\!]$ *and is defined by the clauses:*

i) $[\![x^S]\!]$ *is the identity on* $[\![S]\!]$*, if* x^S *is a variable of sort* S*;*

ii) *Given* $[\![t_i]\!] : [\![FV(t_i)]\!] \to [\![S_i]\!]$ *for* $i = 1, \ldots, n$ *and a function symbol* $(f : S_1, \ldots, S_n \to S)$ *of the language,* $[\![f(t_1, \ldots, t_n)]\!]$ *is the map*

$$[\![FV(f(t_1, \ldots, t_n))]\!] \xrightarrow{(\tilde{t}_i | i=1,\ldots,n)} \textstyle\prod_{i=1}^n [\![S_i]\!] \xrightarrow{[\![f]\!]} [\![S]\!]$$

where $\tilde{t}_i$ *is the composite*

$$[\![FV(f(t_1, \ldots, t_n))]\!] \xrightarrow{\pi_i} [\![FV(t_i)]\!] \xrightarrow{[\![t_i]\!]} [\![S_i]\!]$$

in which π_i *is the appropriate projection, corresponding to the inclusion* $FV(t_i) \subseteq FV(f(t_1, \ldots, t_n))$*.*

Finally, we interpret formulas φ as *subobjects* $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$. You should try to keep in mind the intuition that $[\![\varphi(x_1,\ldots,x_n)]\!]$ is the "subset"

$$\{(a_1,\ldots,a_n)\in A_1\times\cdots\times A_n \mid \varphi[a_1,\ldots,a_n]\}$$

Definition 4.7 *The interpretation $[\![\varphi]\!]$ as subobject of $[\![FV(\varphi)]\!]$ is defined as follows:*

i) $[\![\top]\!]$ *is the maximal subobject of* $[\![FV(\top)]\!]=1$*;*

ii) $[\![t=s]\!]\to[\![FV(t=s)]\!]$ *is the equalizer of*

$$[\![FV(t=s)]\!] \rightrightarrows \begin{matrix}[\![FV(t)]\!]\\ [\![FV(s)]\!]\end{matrix} \begin{matrix}\xrightarrow{[\![t]\!]}\\ \xrightarrow[{[\![s]\!]}]{}\end{matrix} [\![T]\!]$$

if t and s are of sort T; again, the left hand side maps are projections, corresponding to the inclusions of $FV(t)$ and $FV(s)$ into $FV(t=s)$;

iii) *For $(R\subseteq S_1,\ldots,S_m)$ a relation symbol and terms $t_1,\ldots,t_m$ of sorts $S_1,\ldots,S_m$ respectively, let $\bar{t}:[\![FV(R(t_1,\ldots,t_m))]\!]\to\prod_{i=1}^m[\![S_i]\!]$ be the composite map*

$$[\![FV(R(t_1,\ldots,t_m))]\!]\to\prod_{i=1}^m[\![FV(t_i)]\!]\stackrel{\prod_{i=1}^m[\![t_i]\!]}{\rightarrow}\prod_{i=1}^m[\![S_i]\!]$$

Then $[\![R(t_1,\ldots,t_m)]\!]\to[\![FV(R(t_1,\ldots,t_m))]\!]$ is the subobject $(\bar{t})^*([\![R]\!])$, defined by pullback along $\bar{t}$.

iv) *if $[\![\varphi]\!]\to[\![FV(\varphi)]\!]$ and $[\![\psi]\!]\to[\![FV(\psi)]\!]$ are given and*

$$\begin{tikzcd}
{[\![FV(\varphi\wedge\psi)]\!]} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & {[\![FV(\varphi)]\!]} \\
 & {[\![FV(\psi)]\!]}
\end{tikzcd}$$

are again the suitable projections, then $[\![(\varphi\wedge\psi)]\!]\to[\![FV(\varphi\wedge\psi)]\!]$ is the greatest lower bound in $\mathrm{Sub}([\![FV(\varphi\wedge\psi)]\!])$ *of $\pi_1^*([\![\varphi]\!])$ and $\pi_2^*([\![\psi]\!])$;*

v) *if $[\![\varphi]\!]\to[\![FV(\varphi)]\!]$ is given and $\pi:[\![FV(\varphi)]\!]\to[\![FV(\exists x\varphi)]\!]$ the projection, let $[\![FV'(\varphi)]\!]$ be the product of the interpretations of the sorts of the variables in $FV(\varphi)\cup\{x\}$ (so $[\![FV'(\varphi)]\!]=[\![FV(\varphi)]\!]$ if x occurs freely in φ; and $[\![FV'(\varphi)]\!]=[\![FV(\varphi)]\!]\times[\![S]\!]$ if $x=x^S$ does not occur free in φ). Write $\pi':[\![FV'(\varphi)]\!]\to[\![FV(\varphi)]\!]$.*

Now take $[\![\exists x\varphi]\!]\to[\![FV(\exists x\varphi)]\!]$ to be the image of the composition:

$$(\pi')^*([\![\varphi]\!])\to[\![FV'(\varphi)]\!]\stackrel{\pi\pi'}{\to}[\![FV(\exists x\varphi)]\!]$$

We have now given an interpretation of formulas. Basically, a formula φ is *true* under this interpretation if $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ is the maximal subobject; but since we formulate the logic in terms of sequents we rather define when a sequent is true under the interpretation.

Definition 4.8 *A* labelled sequent *is an expression of the form $\psi \vdash_\sigma \varphi$ or $\vdash_\sigma \varphi$ where ψ and φ are the formulas of the sequent (but ψ may be absent), and σ is a finite set of variables which includes all the variables which occur free in a formula of the sequent.*

Let $[\![\sigma]\!] = [\![S_1]\!] \times \cdots \times [\![S_n]\!]$ if $\sigma = \{x_1^{S_1}, \ldots, x_n^{S_n}\}$; there are projections $[\![\sigma]\!] \stackrel{\pi_\varphi}{\to} [\![FV(\varphi)]\!]$ and (in case ψ is there) $[\![\sigma]\!] \stackrel{\pi_\psi}{\to} [\![FV(\psi)]\!]$; we say that the sequent $\psi \vdash_\sigma \varphi$ is true for the interpretation if $(\pi_\psi)^*([\![\psi]\!]) \leq (\pi_\varphi)^*([\![\varphi]\!])$ as subobjects of $[\![\sigma]\!]$, and $\vdash_\sigma \varphi$ is true if $(\pi_\varphi)^*([\![\varphi]\!])$ is the maximal subobject of $[\![\sigma]\!]$.

We also say that φ is true if $\vdash_{\mathrm{FV}(\varphi)} \varphi$ is true.

Exercise 70 Show that the sequent $\vdash \exists x^S(x^S = x^S)$ is true if and only if the unique map $[\![S]\!] \to 1$ is a regular epimorphism. What about the sequent $\vdash_S \top$?

We now turn to the logic. Instead of giving deduction rules and axioms, I formulate a list of closure conditions which determine what sets of labelled sequents will be called a *theory.* I write $\vdash_x$ for $\vdash_{\{x\}}$ and $\vdash$ for $\vdash_\emptyset$.

Definition 4.9 *Given a language, a set T of labelled sequents of that language is called a* theory *iff the following conditions hold (the use of brackets around ψ caters in a, I hope, self-explanatory way for the case distiction as to whether ψ is or is not present):*

i) $\vdash \top$ *is in* T*;*
$\vdash_x x = x$ *is in T for every variable x;*
$x = y \vdash_{\{x,y\}} y = x$ *is in T for variables x, y of the same sort;*
$x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$ *is in T for variables x, y, z of the same sort;*
$R(x_1, \ldots, x_m) \vdash_{\{x_1,\ldots,x_m\}} R(x_1, \ldots, x_m)$ *is in T;*

ii) if $(\psi) \vdash_\sigma \varphi$ is in T then $(\psi) \vdash_\tau \varphi$ is in T whenever $\sigma \subseteq \tau$;

iii) if $(\psi) \vdash_\sigma \varphi$ is in T and $FV(\chi) \subseteq \sigma$ then $(\psi\wedge)\chi \vdash_\sigma \varphi$ and $\chi(\wedge\psi) \vdash_\sigma \varphi$ are in T;

iv) if $(\psi) \vdash_\sigma \varphi$ and $(\psi) \vdash_\sigma \chi$ are in T then $(\psi) \vdash_\sigma \varphi \wedge \chi$ and $(\psi) \vdash_\sigma \chi \wedge \varphi$ are in T;

v) if $\psi \vdash_\sigma \varphi$ is in T and x is a variable not occurring in φ then $\exists x\psi \vdash_{\sigma\setminus\{x\}} \varphi$ is in T;

vi) if x occurs in φ and $(\psi) \vdash_\sigma \varphi[t/x]$ is in T then $(\psi) \vdash_\sigma \exists x\varphi$ is in T;
if x does not occur in φ and $(\psi) \vdash_\sigma \varphi$ and $(\psi) \vdash_\sigma \exists x(x = x)$ are in T, then $(\psi) \vdash_\sigma \exists x\varphi$ is in T;

vii) *if* $(\psi) \vdash_\sigma \varphi$ *is in* T *then* $(\psi[t/x]) \vdash_{\sigma\setminus\{x\}\cup FV(t)} \varphi[t/x]$ *is in* T*;*

viii) *if* $(\psi) \vdash_\sigma \varphi[t/x]$ *and* $(\psi) \vdash_\sigma t = s$ *are in* T *then* $(\psi) \vdash_\sigma \varphi[s/x]$ *is in* T*;*

ix) *if* $(\psi) \vdash_\sigma \varphi$ *and* $\varphi \vdash_\sigma \chi$ *are in* T *then* $(\psi) \vdash_\sigma \chi$ *is in* T

Exercise 71 Show that the sequent $\varphi \vdash_{FV(\varphi)} \varphi$ is in every theory, for every formula φ of the language.

As said, the definition of a theory is a list of closure conditions: every item can be seen as a rule, and a theory is a set of sequents closed under every rule. Therefore, the intersection of any collection of theories is again a theory, and it makes sense to speak, given a set of sequents S, of the theory $Cn(S)$ *generated by* S:

$$Cn(S) = \bigcap\{T | T \text{ is a theory and } S \subseteq T\}$$

We have the following theorem:

Theorem 4.10 (Soundness theorem) *Suppose* $T = Cn(S)$ *and all sequents of* S *are true under the interpretation in the category* $\mathcal{C}$*. Then all sequents of* T *are true under that interpretation.*

Before embarking on the proof, first a lemma:

Lemma 4.11 *Suppose* t *is substitutable for* x *in* φ*. There is an obvious map*

$$[t] : [\![\, FV(\varphi) \setminus \{x\} \cup FV(t) \,]\!] = [\![\, FV(\varphi[t/x]) \,]\!] \to [\![\, FV(\varphi) \,]\!]$$

induced by $[\![\, t \,]\!]$*; the components of* $[t]$ *are projections except for the factor of* $[\![\, \varphi \,]\!]$ *corresponding to* x*, where it is*

$$[\![\, FV(\varphi[t/x]) \,]\!] \to [\![\, FV(t) \,]\!] \stackrel{[\![\, t \,]\!]}{\to} [\![\, \{x\} \,]\!]$$

There is a pullback diagram:

$$\begin{tikzcd}
[\![\, \varphi[t/x] \,]\!] \arrow[r] \arrow[d] & [\![\, FV(\varphi[t/x]) \,]\!] \arrow[d, "{[t]}"] \\
[\![\, \varphi \,]\!] \arrow[r] & [\![\, FV(\varphi) \,]\!]
\end{tikzcd}$$

Exercise 72 Prove this lemma [not trivial. Use induction on φ and proposition 4.4].

Proof. (of theorem 4.10) The proof checks that for every rule in the list of definition 4.9, if the premiss is true then the conclusion is true; in other words, that the set of true sequents is a theory.
i) $\vdash \top$ is true by the definition $[\![\, \top \,]\!] = 1$;
$[\![\, x^S = x^S \,]\!]$ is the equalizer of two maps which are both the identity on $[\![\, S \,]\!]$,

so isomorphic to $[\![S]\!]$. For $x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$, just observe that $E_{12} \wedge E_{23} \leq E_{13}$ if E_{ij} is the equalizer of the two projections $\pi_i, \pi_j : [\![S]\!] \times [\![S]\!] \times [\![S]\!] \to [\![S]\!]$.
ii) This is because if $\sigma \subseteq \tau$ and $\rho : [\![\tau]\!] \to [\![\sigma]\!]$ is the projection, ρ^* is monotone.
iii)-iv) By the interpretation of $\wedge$ as the greatest lower bound of two subobjects, and proposition 4.4.
v) Let

$$\begin{tikzcd}
[\![\sigma]\!] \arrow[r, "\pi"] \arrow[d, "\mu"'] & [\![\sigma \setminus \{x\}]\!] \arrow[r, "\rho"] \arrow[d, "\nu"] & [\![FV(\varphi)]\!] \\
[\![FV(\psi)]\!] & [\![FV(\exists x\psi)]\!] &
\end{tikzcd}$$

the projections. Since by assumption $\mu^*([\![\psi]\!]) \leq (\rho\pi)^*([\![\varphi]\!])$ there is a commutative diagram

$$\begin{tikzcd}
\mu^*([\![\psi]\!]) \arrow[r] \arrow[d] & [\![\sigma]\!] \arrow[d, "\pi"] \\
\rho^*([\![\varphi]\!]) \arrow[r] & [\![\sigma \setminus \{x\}]\!]
\end{tikzcd}$$

By proposition 4.4, $\nu^*([\![\exists x\psi]\!])$ is the image of the map $\mu^*([\![\psi]\!]) \to [\![\sigma \setminus \{x\}]\!]$, so $\nu^*([\![\psi]\!]) \leq \rho^*([\![\varphi]\!])$ in $\mathrm{Sub}([\![\sigma \setminus \{x\}]\!])$.
vi) Suppose x occurs free in φ. Consider the commutative diagram

$$\begin{tikzcd}
 & [\![\sigma]\!] \arrow[dl, "\pi"'] \arrow[d, "\pi'"] \arrow[drrr, "\pi''"] & & & \\
[\![FV(\psi)]\!] & [\![FV(\varphi[t/x])]\!] \arrow[r, "{[t]}"] & [\![FV(\varphi)]\!] \arrow[rr, "\rho"] & & [\![FV(\varphi) \setminus \{x\}]\!]
\end{tikzcd}$$

with $[t]$ as in lemma 4.11 and the other maps projections. Now $[\![\varphi]\!] \leq \rho^*([\![\exists x\varphi]\!])$ because $[\![\varphi]\!] \to [\![FV(\varphi)]\!] \xrightarrow{\rho} [\![FV(\varphi) \setminus \{x\}]\!]$ factors through $[\![\exists x\varphi]\!]$ by definition; so if $\pi^*([\![\psi]\!]) \leq \pi'^*([\![\varphi[t/x]]\!])$ then with lemma 4.11,

$$\pi^*([\![\psi]\!]) \leq \pi'^*([\![\varphi[t/x]]\!]) = \pi'^*[t]^*([\![\varphi]\!]) \leq \pi'^*[t]^*\rho^*([\![\exists x\varphi]\!]) = \pi''^*([\![\exists x\varphi]\!])$$

in $\mathrm{Sub}([\![\sigma]\!])$ and we are done.

The case of x not occurring in φ is left to you.
vii) Direct application of lemma 4.11
viii-ix) Left to you. ∎

Exercise 73 Fill in the "left to you" gaps in the proof.

4.3 The language $\mathcal{L}(\mathcal{C})$ and theory $T(\mathcal{C})$ associated to a regular category $\mathcal{C}$

Given a regular category $\mathcal{C}$ (which, to be precise, must be assumed to be small), we associate to $\mathcal{C}$ the language which has a sort C for every object of $\mathcal{C}$, and a function symbol $(f : C \to D)$ for every arrow $f : C \to D$ of $\mathcal{C}$.

This language is called $\mathcal{L}(\mathcal{C})$ and it has trivially an interpretation in $\mathcal{C}$.

The theory $T(\mathcal{C})$ is the set of sequents of $\mathcal{L}(\mathcal{C})$ which are true for this interpretation.

One of the points of categorical logic is now, that categorical statements about objects and arrows in $\mathcal{C}$ can be reformulated as statements about the truth of certain sequents in $\mathcal{L}(\mathcal{C})$. You should read the relevant sequents as expressing that we can "do as if the category were Set".

Examples

a) C is a terminal object of $\mathcal{C}$ if and only if the sequents $\vdash_{x,y} x = y$ and $\vdash \exists x(x = x)$ are valid, where x, y variables of sort C;

b) the arrow $f : A \to B$ is mono in $\mathcal{C}$ if and only if the sequent $f(x) = f(y) \vdash_{x,y} x = y$ is true;

c) The square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "d"'] & D \end{tikzcd}$$

is a pullback square in $\mathcal{C}$ if and only if the sequents

$$h(x^B) = d(y^C) \vdash_{x,y} \exists z^A(f(z) = x \wedge g(z) = y)$$

and

$$f(z^A) = f(z'^A) \wedge g(z^A) = g(z'^A) \vdash_{z,z'} z = z'$$

are true;

d) the fact that $f : A \to B$ is epi can not similarly be expressed! But: f is regular epi if and only if

$$\vdash_{x^B} \exists y^A(f(y) = x)$$

is true;

e) $A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$ is an equalizer diagram iff f is mono (see b) and the sequent

$$g(x^B) = h(x^B) \vdash_{x^B} \exists y^A(f(y) = x)$$

is true.

Exercise 74 Check (a number of) these statements. Give the sequent(s) corresponding to the statement that $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$ is a product diagram.

Exercise 75 Check that the formulas $\exists x\varphi$ and $\exists x(x = x \wedge \varphi)$ are *equivalent*, that is, every theory contains the sequents

$$\exists x\varphi \vdash_\sigma \exists x(x = x \wedge \varphi)$$

and

$$\exists x(x = x \wedge \varphi) \vdash_\sigma \exists x\varphi$$

for any σ containing the free variables of $\exists x\varphi$.

Exercise 76 Suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "d"'] & D \end{tikzcd}$$

is a pullback diagram in a regular category $\mathcal{C}$, with h regular epi. Use regular logic to show: if f is mono, then so is d. Give also a categorical proof of this fact, and compare the two proofs.

Exercise 77 Can you express: $A \xrightarrow{f} B$ is regular mono? [Hint: don't waste too much time in trying!]

4.4 The category $\mathcal{C}(T)$ associated to a theory T: Completeness Theorem

The counterpart of Theorem 4.10 (the Soundness Theorem) is of course a completeness theorem: suppose that the sequent $(\psi) \vdash_\sigma \varphi$ is true in every interpretation which makes all sequents from T true. We want to conclude that $(\psi) \vdash_\sigma \varphi$ is in T.

To do this, we build a category $\mathcal{C}(T)$, a so-called *syntactic category*, which will be regular, and which allows an interpretation $[\![\cdot]\!]$ such that exactly the sequents in T will be true for $[\![\cdot]\!]$.

The construction of $\mathcal{C}(T)$ is as follows. *Objects* are formulas of the language $\mathcal{L}$, up to renaming of free variables; so if φ is a formula in distinct free variables $x_1, \ldots, x_n$ and $v_1, \ldots, v_n$ are distinct variables of matching sorts, then φ and $\varphi(v_1/x_1, \ldots, v_n/x_n)$ are the same object. Because of this stipulation, if I define morphisms between $\varphi(\vec{x})$ and $\psi(\vec{y})$, I can always assume that the collections of variables $\vec{x}$ and $\vec{y}$ are disjoint (even when treating a morphism from φ to itself, I take φ and a renaming $\varphi(\vec{x'})$). Given $\varphi(\vec{x})$ and $\psi(\vec{y})$ (where $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_m)$), a *functional relation* from φ to ψ is a formula $\chi(\vec{x}, \vec{y})$ such that the sequents

$$\chi(\vec{x}, \vec{y}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m\}} \varphi(\vec{x}) \wedge \psi(\vec{y})$$
$$\chi(\vec{x}, \vec{y}) \wedge \chi(\vec{x}, \vec{y'}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m,y'_1,\ldots,y'_m\}} y_1 = y'_1 \wedge \cdots \wedge y_m = y'_m$$
$$\varphi(\vec{x}) \vdash_{\{x_1,\ldots,x_n\}} \exists y_1 \cdots \exists y_m \, \chi(\vec{x}, \vec{y})$$

are all in T. If $FV(\psi) = \emptyset$, the second requirement is taken to be vacuous, i.e. trivially fulfilled.

A *morphism* from φ to ψ is an equivalence class $[\chi]$ of functional relations from φ to ψ, where χ_1 and χ_2 are equivalent iff the sequents

$$\begin{array}{c} \chi_1(\vec{x},\vec{y}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m\}} \chi_2(\vec{x},\vec{y}) \\ \chi_2(\vec{x},\vec{y}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m\}} \chi_1(\vec{x},\vec{y}) \end{array}$$

are in T (in fact, given that χ_1 and χ_2 are functional relations, one of these sequents is in T iff the other is).

Composition of morphisms is defined as follows: if $\chi_1(\vec{x},\vec{y})$ represents a morphism $\varphi \to \psi$ and $\chi_2(\vec{y},\vec{z})$ a morphism $\psi \to \omega$, the composition $[\chi_2] \circ [\chi_1] : \varphi \to \omega$ is represented by the functional relation

$$\chi_{21} \equiv \exists y_1 \cdots \exists y_m \, (\chi_1(\vec{x},\vec{y}) \wedge \chi_2(\vec{y},\vec{z}))$$

Exercise 78 Show:

a) χ_{21} is a functional relation from φ to ω;

b) the class of χ_{21} does not depend on the choice of representatives χ_1 and χ_2;

c) composition is associative.

The *identity* arrow from $\varphi(\vec{x})$ to itself (i.e. to $\varphi(\vec{x'})$, given our renaming convention), is represented by the formula

$$\varphi(\vec{x}) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$$

Exercise 79 Show that this definition is correct: that this formula is a functional relation, and defines an identity arrow.

We have defined the category $\mathcal{C}(T)$.

Theorem 4.12 *$\mathcal{C}(T)$ is a regular category.*

Proof. The formula $\top$ is a terminal object in $\mathcal{C}(T)$: for every formula φ, φ itself represents the unique morphism $\varphi \to \top$. Given formulas $\varphi(\vec{x})$ and $\psi(\vec{y})$, the formula $\varphi \wedge \psi$ is a product, with projections $\varphi \wedge \psi \to \varphi(\vec{x'})$ (renaming!) represented by the formula

$$\varphi(\vec{x}) \wedge \psi(\vec{y}) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$$

and $\varphi \wedge \psi$ to $\psi(\vec{y'})$ similarly defined.

If $\chi_1(\vec{x},\vec{y})$ and $\chi_2(\vec{x},\vec{y})$ represent morphisms $\varphi(\vec{x}) \to \psi(\vec{y})$ let $\omega(\vec{x})$ be the formula

$$\exists y_1 \cdots \exists y_m (\chi_1(\vec{x},\vec{y}) \wedge \chi_2(\vec{x},\vec{y}))$$

Then $\omega(\vec{x}) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$ represents a morphism $\omega \to \varphi(\vec{x'})$ which is the equalizer of $[\chi_1]$ and $[\chi_2]$.

This takes care of finite limits.

Now if $[\chi_1(\vec{x},\vec{y})] : \varphi(\vec{x}) \to \psi_1(\vec{y})$ and $[\chi_2(\vec{x},\vec{y})] : \varphi(\vec{x}) \to \psi_2(\vec{y})$, then you can check that $[\chi_2]$ coequalizes the kernel pair of $[\chi_1]$ if and only if the sequent

$$\exists \vec{y}\, (\chi_1(\vec{x},\vec{y}) \wedge \chi_1(\vec{x'},\vec{y})) \vdash_{\{\vec{x},\vec{x'}\}} \exists \vec{v}(\chi_2(\vec{x},\vec{v}) \wedge \chi_2(\vec{x'},\vec{v}))$$

is in T. This is the case if and only if $[\chi_2]$ factors through the obvious map from φ to $\exists \vec{x}\chi_1(\vec{x},\vec{y})$ which is therefore the image of $[\chi_1]$, i.e. the coequalizer of its kernel pair.

We see at once that $[\chi] : \varphi \to \psi$ is regular epi iff the sequent

$$\psi(\vec{y}) \vdash_{\{\vec{y}\}} \exists \vec{x}\, \chi(\vec{x},\vec{y})$$

is in T; using the description of finite limits one easily checks that these are stable under pullback ∎

We define $[\![\cdot]\!]$, the standard interpretation of $\mathcal{L}$ in $\mathcal{C}(T)$, as follows:

- the interpretation $[\![S]\!]$ of a sort S is the formula $x = x$, where x is a variable of sort S;
- if $(f : S_1, \ldots, S_n \to S)$ is a function symbol of $\mathcal{L}$, $[\![f]\!]$ is the functional relation
$$f(x_1, \ldots, x_n) = x$$
- if $(R \subseteq S_1, \ldots, S_n)$ is a relation symbol of $\mathcal{L}$, the subobject $[\![R]\!]$ of $[\![S_1]\!] \times \cdots \times [\![S_n]\!]$ is represented by the mono
$$R(x_1, \ldots, x_n) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$$

I now state the important facts about $\mathcal{C}(T)$ and $[\![\cdot]\!]$ as nontrivial exercises. Here we say that, given the theory T, formulas φ and ψ (in the same free variables $x_1, \ldots, x_n$ are *equivalent in* T if the sequents $\varphi \vdash_{\{x_1,\ldots,x_n\}} \psi$ and $\psi \vdash_{\{x_1,\ldots,x_n\}} \varphi$ are both in T.

Exercise 80 If t is a term of sort S in $\mathcal{L}$, in variables $x_1^{S_1}, \ldots, x_n^{S_n}$, the functional relation from $[\![S_1]\!] \times \cdots \times [\![S_n]\!] \equiv x_1 = x_1 \wedge \cdots \wedge x_n = x_n$, to $[\![S]\!] \equiv x = x$, representing $[\![t]\!]$, is equivalent in T to the formula

$$t(x_1, \ldots, x_n) = x$$

Exercise 81 If φ is a formula of $\mathcal{L}$ in free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, the subobject $[\![\varphi]\!] \to [\![S_1]\!] \times \cdots \times [\![S_n]\!]$ is represented by a functional relation χ from a formula ψ to $x_1 = x_1 \wedge \cdots \wedge x_n = x_n$, such that:

a) ψ is a formula in variables $x_1', \ldots, x_n'$ and $\psi(x_1', \ldots, x_n')$ is equivalent in T to $\varphi(x_1', \ldots, x_n')$;

b) $\chi(x'_1, \ldots, x'_n, x_1, \ldots, x_n)$ is equivalent in T to

$$\varphi(x'_1, \ldots, x'_n) \wedge x'_1 = x_1 \wedge \cdots \wedge x'_n = x_n$$

Exercise 82 The sequent $(\psi) \vdash_\sigma \varphi$ is true in the interpretation $[\![\cdot]\!]$ if and only if $(\psi) \vdash_\sigma \varphi$ is in T.

Exercise 83 Let $\mathcal{E}$ be a regular category and T a theory. Call a functor between regular categories *regular* if it preserves finite limits and regular epis.

Then every regular functor: $\mathcal{C}(T) \to \mathcal{E}$ gives rise to an interpretation of the language of T in $\mathcal{E}$, which makes all sequents in T true.

Conversely, given such an interpretation of T in $\mathcal{E}$, there is, up to natural isomorphism, a unique regular functor: $\mathcal{C}(T) \to \mathcal{E}$ with the property that it maps the standard interpretation of T in $\mathcal{C}(T)$ to the given one in $\mathcal{E}$.

Exercise 84 This exercise constructs the "free regular category on a given category $\mathcal{C}$". Given $\mathcal{C}$, which is not assumed to be regular (or to have finite limits), let $\mathcal{L}$ be the language of $\mathcal{C}$ as before: it has a sort C for every object C of $\mathcal{C}$, and a function symbol $(f : C \to D)$ for every arrow of $\mathcal{C}$. Let T be the theory generated by the following set S of sequents: for every identity arrow $i : C \to C$ in $\mathcal{C}$, the sequent $\vdash_x x = i(x)$ is in S; and for every composition $f = gh$ of arrows in $\mathcal{C}$, the sequent $\vdash_x f(x) = g(h(x))$ is in S.

a) Show that interpretations of $\mathcal{L}$ in a regular category $\mathcal{E}$ which make all sequents of S true, correspond bijectively to functors from $\mathcal{C}$ to $\mathcal{E}$.

b) Show that there is a functor $\eta : \mathcal{C} \to \mathcal{C}(T)$ such that for every functor $F : \mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ regular, there is, up to isomorphism, a unique regular functor $\tilde{F} : \mathcal{C}(T) \to \mathcal{E}$ such that $\tilde{F}\eta = F$. $\mathcal{C}(T)$ is the *free regular category on $\mathcal{C}$*.

c) Show that if, in this situation, $\mathcal{C}$ has finite limits, η does not preserve them! How would one construct the "free regular category on $\mathcal{C}$, preserving the limits which exist in $\mathcal{C}$"?

4.5 Example of a regular category

In this section, I treat an example of a type of regular categories which are important in categorical logic. They are categories of Ω-valued sets for some frame Ω. Let's define some things.

Definition 4.13 *A* frame Ω *is a partially ordered set which has suprema (least upper bounds)* $\bigvee B$ *of all subsets* B*, and infima (meets)* $\bigwedge A$ *for* finite *subsets* A *(so, there is a top element* $\top$ *and every pair of elements* x, y *has a meet* $x \wedge y$*), and moreover,* $\wedge$ *distributes over* $\bigvee$*, that is,*

$$x \wedge \bigvee B = \bigvee \{x \wedge b | b \in B\}$$

for $x \in \Omega$*,* $B \subseteq \Omega$*.*

Given a frame Ω we define the category $\mathcal{C}_\Omega$ as follows:
Objects are functions $X \stackrel{E_X}{\to} \Omega$, X a set;
Maps from (X, E_X) to (Y, E_Y) are functions $X \stackrel{f}{\to} Y$ satisfying the requirement that $E_X(x) \le E_Y(f(x))$ for all $x \in X$.
It is easily seen that the identity function satisfies this requirement, and if two composable functions satisfy it, their composition does; so we have a category.

Proposition 4.14 *$\mathcal{C}_\Omega$ is a regular category.*

Proof. Let $\{*\}$ be any one-element set, together with the function which sends $*$ to the top element of Ω. Then $\{*\} \to \Omega$ is a terminal object of $\mathcal{C}_\Omega$.

Given (X, E_X) and (Y, E_Y), a product of the two is the object $(X \times Y, E_{X\times Y})$ where $E_{X\times Y}(x, y)$ is defined as $E_X(x) \wedge E_Y(y)$.

Given two arrows $f, g : (X, E_X) \to (Y, E_Y)$ their equalizer is $(X', E_{X'})$ where $X' \subseteq X$ is $\{x \in X | f(x) = g(x)\}$ and $E_{X'}$ is the restriction of E_X to X'.

This is easily checked, and $\mathcal{C}_\Omega$ is a finitely complete category.

An arrow $f : (X, E_X) \to (Y, E_Y)$ is regular epi if and only if the function f is surjective and for all $y \in Y$, $E_Y(y) = \bigvee\{E_X(x) \mid f(x) = y\}$.

For suppose f is such, and $g : (X, E_X) \to (Z, E_Z)$ coequalizes the kernel pair of f. Then $g(x) = g(x')$ whenever $f(x) = f(x')$, and so for all $y \in Y$, since $f(x) = y$ implies $E_X(x) \le E_Z(g(x))$, we have

$$E_Y(y) = \bigvee\{E_X(x)|f(x) = y\} \le E_Z(g(x))$$

so there is a unique map $h : (Y, E_Y) \to (Z, E_Z)$ such that $g = hf$; that is f is the coequalizer of its kernel pair.

The proof of the converse is left to you.

Finally we must show that regular epis are stable under pullback. This is an exercise. ■

Exercise 85 Show that the pullback of
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
(I suppress reference to the E_X etc.) is (up to isomorphism) the set $\{(x, y)|f(x) = g(y)\}$, with $E(x, y) = E_X(x) \wedge E_Y(y)$; and then, use the distributivity of Ω to show that regular epis are stable under pullback.

Exercise 86 Fill in the other gap in the proof: if $f : (X, E_X) \to (Y, E_Y)$ is a regular epi, then f satisfies the condition given in the proof.

Exercise 87 Given $(X, E_X) \stackrel{f}{\to} (Y, E_Y)$, give the interpretation of the formula $\exists x(f(x) = y)$, as subobject of (Y, E_Y).

Exercise 88 Characterize those objects (X, E_X) for which the unique map to the terminal object is a regular epimorphism.

Exercise 89 Give a categorical proof of the statement: if f is the coequalizer of something, it is the coequalizer of its kernel pair.

Exercise 90 Characterize the regular monos in $\mathcal{C}_\Omega$.

Exercise 91 For every element u of Ω, let 1_u be the object of $\mathcal{C}_\Omega$ which is the function from a one-element set into Ω, with value u. Prove that every object of $\mathcal{C}_\Omega$ is a coproduct of objects of form 1_u.

5 Adjunctions

The following kind of problem occurs quite regularly: suppose we have a functor $\mathcal{C} \xrightarrow{G} \mathcal{D}$ and for a given object D of $\mathcal{D}$, we look for an object $G(C)$ which "best approximates" D, in the sense that there is a map $D \xrightarrow{\eta} G(C)$ with the property that any other map $D \xrightarrow{g} G(C')$ factors uniquely as $G(f)\eta$ for $f : C \to C'$ in $\mathcal{C}$.

We have seen, that if G is the inclusion of Abgp into Grp, the abelianization of a group is an example. Another example is the Čech-Stone compactification in topology: for a completely regular topological space X one constructs a compact Hausdorff space βX out of it, and a map $X \to \beta X$, such that any continuous map from X to a compact Hausdorff space factors uniquely through this map.

Of course, what we described here is a sort of "right-sided" approximation; the reader can define for himself what the notion for a left-sided approxiamtion would be.

The categorical description of this kind of phenomena goes via the concept of *adjunction*, which this chapter is about.

5.1 Adjoint functors

Let $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ be a pair of functors between categories $\mathcal{C}$ and $\mathcal{D}$.

We say that F is *left adjoint* to G, or G is *right adjoint* to F, notation: $F \dashv G$, if there is a natural bijection:

$$\mathcal{C}(FD, C) \xrightarrow{m_{D,C}} \mathcal{D}(D, GC)$$

for each pair of objects $C \in \mathcal{C}_0, D \in \mathcal{D}_0$. Two maps $f : FD \to C$ in $\mathcal{C}$ and $g : D \to GC$ in $\mathcal{D}$ which correspond to each other under this correspondence are called *transposes* of each other.

The naturality means that, given $f : D \to D'$, $g : C' \to C$ in $\mathcal{D}$ and $\mathcal{C}$ respectively, the diagram

$$\begin{tikzcd}
\mathcal{C}(FD, C) \arrow[r, "m_{D,C}"] & \mathcal{D}(D, GC) \\
\mathcal{C}(FD', C') \arrow[u, "\mathcal{C}(Ff,g)"] \arrow[r, "m_{D',C'}"'] & \mathcal{D}(D', GC') \arrow[u, "\mathcal{D}(f,Gg)"']
\end{tikzcd}$$

commutes in Set. Remind yourself that given $\alpha : FD' \to C'$, $\mathcal{C}(Ff, g)(\alpha) : FD \to C$ is the composite

$$FD \xrightarrow{Ff} FD' \xrightarrow{\alpha} C' \xrightarrow{g} C$$

Such a family $m = (m_{D,C} | D \in \mathcal{D}_0, C \in \mathcal{C}_0)$ is then completely determined by the values it takes on identities; i.e. the values

$$m_{D,FD}(\mathrm{id}_{FD}) : D \to GFD$$

For, given $\alpha : FD \to C$, since $\alpha = \mathcal{C}(\mathrm{id}_{FD}, \alpha)(\mathrm{id}_{FD})$,

$$m_{D,C}(\alpha) \quad = \quad m_{D,C}(\mathcal{C}(\mathrm{id}_{FD}, \alpha)(\mathrm{id}_{FD})) \quad = \quad \mathcal{D}(\mathrm{id}_D, G(\alpha))(m_{D,FD}(\mathrm{id}_{FD}))$$

which is the composite

$$D \xrightarrow{m_{D,FD}(\mathrm{id}_{FD})} GFD \xrightarrow{\quad G(\alpha) \quad} G(C)$$

The standard notation for $m_{D,FD}(\mathrm{id}_{FD})$ is $\eta_D : D \to GF(D)$.

Exercise 92 Show that $(\eta_D : D \in \mathcal{D}_0)$ is a natural transformation:

$$\mathrm{id}_{\mathcal{D}} \Rightarrow GF$$

By the same reasoning, the natural family $(m^{-1}_{D,C} | D \in \mathcal{D}_0, C \in \mathcal{C}_0)$ is completely determined by its actions on identities

$$m^{-1}_{GC,C}(\mathrm{id}_{GC}) : FGC \to C$$

Again, the family $(m^{-1}_{GC,C}(\mathrm{id}_{GC}) | C \in \mathcal{C}_0)$ is a natural transformation: $FG \Rightarrow \mathrm{id}_{\mathcal{C}}$. We denote its components by ε_C and this is also standard notation.

We have that $m^{-1}_{D,C}(\beta : D \to GC)$ is the composite

$$FD \xrightarrow{F\beta} FGC \xrightarrow{\varepsilon_C} C$$

Now making use of the fact that $m_{D,C}$ and $m^{-1}_{D,C}$ are each others inverse we get that for all $\alpha : FD \to C$ and $\beta : D \to GC$ the diagrams

$$\begin{tikzcd} D \arrow[r, "\beta"] \arrow[d, "\eta_D"'] & GC \\ GFD \arrow[r, "GF(\beta)"'] & GFG(C) \arrow[u, "G(\varepsilon_C)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FD \arrow[r, "\alpha"] \arrow[d, "F(\eta_D)"'] & C \\ FGFD \arrow[r, "FG(\alpha)"'] & FGC \arrow[u, "\varepsilon_C"'] \end{tikzcd}$$

commute; applying this to the identities on FD and GC we find that we have commuting diagrams of natural transformations:

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta\star G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & GFG \arrow[d, Rightarrow, "G\circ\varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, Rightarrow, "F\circ\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon\star F"] \\ & F \end{tikzcd}$$

Here $\eta \star G$ denotes $(\eta_{GC} | C \in \mathcal{C}_0)$ and $G \circ \varepsilon$ denotes $(G(\varepsilon_C) | C \in \mathcal{C}_0)$.

Conversely, given $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ with natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ which satisfy the above *triangle equalities*, we have that $F \dashv G$.

The tuple $(F, G, \varepsilon, \eta)$ is called an *adjunction*. η is the *unit* of the adjunction, ε the *counit*.

Exercise 93 Prove the last statement, i.e. given $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ satisfying $(G \circ \varepsilon) \cdot (\eta \star G) = \mathrm{id}_G$ and $(\varepsilon \star F) \cdot (F \circ \eta) = \mathrm{id}_F$, we have $F \dashv G$.

Exercise 94 Given $\mathcal{C} \underset{G_1}{\overset{F_1}{\leftrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\leftrightarrows}} \mathcal{E}$, if $F_1 \dashv G_1$ and $F_2 \dashv G_2$ then $F_1F_2 \dashv G_2G_1$.

Examples. The world is full of examples of adjoint functors. We have met several:

a) Consider the forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ and the free functor $F : \mathrm{Set} \to \mathrm{Grp}$. Given a function from a set A to a group G (which is an arrow $A \to U(G)$ in Set) we can uniquely extend it to a group homomorphism from $(\tilde{A}, \star)$ to G (see example e) of 1.1), i.e. an arrow $F(A) \to G$ in Grp, and conversely. This is natural in A and G, so $F \dashv U$;

b) The functor $\mathrm{Dgrph} \to \mathrm{Cat}$ given in example f) of 1.1 is left adjoint to the forgetful functor $\mathrm{Cat} \to \mathrm{Dgrph}$;

c) Given functors $P \underset{G}{\overset{F}{\leftrightarrows}} Q$ between two preorders P and Q, $F \dashv G$ if and only if we have the equivalence

$$y \leq G(x) \Leftrightarrow F(y) \leq x$$

for $x \in P, y \in Q$; if and only if we have $FG(x) \leq x$ and $y \leq GF(y)$ for all x, y;

d) In example m) of 1.1 we did "abelianization" of a group G. We made use of the fact that any homomorphism $G \to H$ with H abelian, factors uniquely through $G/[G,G]$, giving a natural 1-1 correspondence

$$\mathrm{Grp}(G, I(H)) \xrightarrow{\sim} \mathrm{Abgp}(G/[G,G], H)$$

where $I : \mathrm{Abgp} \to \mathrm{Grp}$ is the inclusion. So abelianization is left adjoint to the inclusion functor of abelian groups into groups;

e) The *free monoid* $F(A)$ on a set A is just the set of strings on the alphabet A. $F : \mathrm{Set} \to \mathrm{Mon}$ is a functor left adjoint to the forgetful functor from Mon to Set;

f) Given a set X we have seen (example g) of 2.2) the product functor $(-) \times X : \mathrm{Set} \to \mathrm{Set}$, assigning the product $Y \times X$ to a set Y.

   Since there is a natural bijection between functions $Y \times X \to Z$ and functions $Y \to Z^X$, the functor $(-)^X : \mathrm{Set} \to \mathrm{Set}$ is right adjoint to $(-) \times X$;

g) Example e) of 2.2 gives two functors $F, G : \mathrm{Set} \to \mathrm{Cat}$. F and G are respectively left and right adjoint to the functor $\mathrm{Cat} \stackrel{\mathrm{Ob}}{\to} \mathrm{Set}$ which assigns to a (small) category its set of objects (to be precise, for this example to work we have to take for Cat the category of *small* categories), and to a functor its action on objects.

h) Given a regular category $\mathcal{C}$ we saw in 4.1 that every arrow $f : X \to Y$ can be factored as a regular epi followed by a monomorphism. In Exercise 69 you were asked to show that there is a function $\exists_f : \mathrm{SUB}(X) \to \mathrm{Sub}(Y)$ such that the equivalence $\exists_f(M) \leq N \Leftrightarrow M \leq f^*(N)$ holds, for arbitrary subobjects M and N of X and Y, respectively.

But this just means thet $\exists_f \dashv f^*$

We can also express this logically: in the logic developed in chapter 4, for any formulas $M(x)$ and $N(y)$, the sequents

$$\exists x(f(x) = y \wedge M(x)) \vdash_y N(y)$$

and

$$M(x) \vdash_x N(f(x))$$

are equivalent.

One of the slogans of categorical logic is therefore, that "existential quantification is left adjoint to substitution".

i) Let $\mathcal{C}$ be a category with finite products; for $C \in \mathcal{C}_0$ consider the slice category $\mathcal{C}/C$. There is a functor $C^* : \mathcal{C} \to \mathcal{C}/C$ which assigns to D the object $C \times D \stackrel{\pi_C}{\to} C$ of $\mathcal{C}/C$, and to maps $D \stackrel{f}{\to} D'$ the map $\mathrm{id}_C \times f$.

C^* has a left adjoint Σ_C which takes the domain: $\Sigma_C(D \to C) = D$.

j) Let $P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ be the functor which takes the powerset on objects, and for $X \stackrel{f}{\to} Y$, $P(f) : P(Y) \to P(X)$ gives for each subset B of Y its inverse image under f.

Now P might as well be regarded as a functor $\mathrm{Set} \to \mathrm{Set}^{\mathrm{op}}$; let's write $\bar{P}$ for that functor. Since there is a natural bijection:

$$\mathrm{Set}(X, P(Y)) \stackrel{\sim}{\to} \mathrm{Set}(Y, P(X)) = \mathrm{Set}^{\mathrm{op}}(\bar{P}(X), Y)$$

we have an adjunction $\bar{P} \dashv P$.

Exercise 95 A general converse to the last example. Suppose that $F : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ is a functor, such that for the corresponding functor $\bar{F} : \mathrm{Set} \to \mathrm{Set}^{\mathrm{op}}$ we have that $\bar{F} \dashv F$. Then there is a set A such that F is naturally isomorphic to $\mathrm{Set}(-, A)$.

Exercise 96 Suppose that $\mathcal{C} \xleftarrow{F} \mathcal{D}$ is a functor and that for each object C of $\mathcal{C}$ there is an object GC of $\mathcal{D}$ and an arrow $\varepsilon_C : FGC \to C$ with the property that for every object D of $\mathcal{D}$ and any map $FD \xrightarrow{f} C$, there is a unique $\tilde{f} : D \to GC$ such that

$$\begin{tikzcd} FD \arrow[rr, "f"] \arrow[dr, "F\tilde{f}"'] & & C \\ & FGC \arrow[ur, "\varepsilon_C"'] & \end{tikzcd}$$

commutes.

Prove that $G : \mathcal{C}_0 \to \mathcal{D}_0$ extends to a functor $G : \mathcal{C} \to \mathcal{D}$ which is right adjoint to F, and that $(\varepsilon_C : FGC \to C | C \in \mathcal{C}_0)$ is the counit of the adjunction.

Construct also the unit of the adjunction.

Exercise 97 Given $\mathcal{C} \xrightarrow{G} \mathcal{D}$, for each object D of $\mathcal{D}$ we let $(D{\downarrow}G)$ denote the category which has as objects pairs (C, g) where C is an object in $\mathcal{C}$ and $g : D \to GC$ is an arrow in $\mathcal{D}$. An arrow $(C, g) \to (C', g')$ in $(D{\downarrow}G)$ is an arrow $f : C \to C'$ in $\mathcal{C}$ which makes

$$\begin{tikzcd} & D \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ GC \arrow[rr, "Gf"'] & & GC' \end{tikzcd}$$

commute.

Show that G has a left adjoint if and only if for each D, the category $(D{\downarrow}G)$ has an initial object.

Exercise 98 (Uniqueness of adjoints) Any two left (or right) adjoints of a given functor are isomorphic as objects of the appropriate functor category.

Exercise 99 $\mathcal{D} \to \mathbf{1}$ has a right adjoint iff $\mathcal{D}$ has a terminal object, and a left adjoint iff $\mathcal{D}$ has an initial object.

Exercise 100 Suppose $\mathcal{D}$ has both an initial and a terminal object; denote by L the functor $\mathcal{D} \to \mathcal{D}$ which sends everything to the initial, and by R the one which sends everything to the terminal object. $L \dashv R$.

Exercise 101 Let $F \dashv G$ with counit $\varepsilon : FG \Rightarrow \mathrm{id}$. Show that ε is a natural isomorphism if and only if G is full and faithful; and G is faithful if and only if all components of ε are epimorphisms.

Exercise 102 Suppose that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories with pseudo inverse $G : \mathcal{D} \to \mathcal{C}$. Show that both $F \dashv G$ and $G \dashv F$ hold.

5.2 Expressing (co)completeness by existence of adjoints; preservation of (co)limits by adjoint functors

Given categories $\mathcal{C}$ and $\mathcal{D}$, we defined for every functor $F : \mathcal{C} \to \mathcal{D}$ its *limit* (or limiting cone), if it existed, as a pair (D, μ) with $\mu : \Delta_D \Rightarrow F$, and (D, μ) terminal in the category of cones for F.

Any other natural transformation $\mu' : \Delta_{D'} \Rightarrow F$ factors uniquely through (D, μ) via an arrow $D' \to D$ in $\mathcal{D}$ and conversely, every arrow $D' \to D$ gives rise to a natural transformation $\mu' : \Delta_{D'} \Rightarrow F$.

So there is a 1-1 correspondence between

$$\mathcal{D}(D', D) \text{ and } \mathcal{D}^{\mathcal{C}}(\Delta_{D'}, F)$$

which is natural in D'.

Since every arrow $D' \to D''$ in $\mathcal{D}$ gives a natural transformation $\Delta_{D'} \Rightarrow \Delta_{D''}$ (example i) of 2.2), there is a functor $\Delta_{(-)} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$.

The above equation now means that:

Proposition 5.1 *$\mathcal{D}$ has all limits of type $\mathcal{C}$ (i.e. every functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ has a limiting cone in $\mathcal{D}$) if and only if $\Delta_{(-)}$ has a right adjoint.*

Exercise 103 Give an exact proof of this proposition.

Exercise 104 Use duality to deduce the dual of the proposition: $\mathcal{D}$ has all colimits of type $\mathcal{C}$ if and only if $\Delta_{(-)} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ has a left adjoint.

A very important aspect of adjoint functors is their behaviour with respect to limits and colimits.

Theorem 5.2 *Let* $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ *such that $F \dashv G$. Then:*

a) G preserves all limits which exist in $\mathcal{C}$;

b) F preserves all colimits which exist in $\mathcal{D}$.

Proof. Suppose $M : \mathcal{E} \to \mathcal{C}$ has a limiting cone (C, μ) in $\mathcal{C}$. Now a cone (D, ν) for GM is a natural family $D \xrightarrow{\nu_E} GM(E)$, i.e. such that

$$\begin{tikzcd} D \arrow[r, "\nu_E"] \arrow[dr, "\nu_{E'}"'] & GM(E) \arrow[d, "GM(e)"] \\ & GM(E') \end{tikzcd}$$

commutes for every $E \xrightarrow{e} E'$ in $\mathcal{E}$.

This transposes under the adjunction to a family $(FD \stackrel{\tilde{\nu}_E}{\to} ME | E \in \mathcal{E}_0)$ and the naturality requirement implies that

$$\begin{tikzcd} FD \arrow[r, "\tilde{\nu}_E"] \arrow[dr, "\tilde{\nu}_{E'}"'] & ME \arrow[d, "M(e)"] \\ & ME' \end{tikzcd}$$

commutes in $\mathcal{C}$, in other words, that (FD, ν) is a cone for M in $\mathcal{C}$. There is, therefore, a unique map of cones from $(FD, \tilde{\nu})$ to (C, μ).

Transposing back again, we get a unique map of cones $(D, \nu) \to (GC, G \circ \mu)$. That is to say that $(GC, G \circ \mu)$ is terminal in Cone(GM), so a limiting cone for GM, which was to be proved.

The argument for the other statement is dual. ∎

Exercise 105 Given $\begin{tikzcd} \mathcal{C} \arrow[r, shift right, "G"'] & \mathcal{D} \arrow[l, shift right, "F"'] \end{tikzcd}$, $F \dashv G$ and $M : \mathcal{E} \to \mathcal{C}$. Show that the functor Cone$(M) \to$ Cone(GM) induced by G has a left adjoint.

From the theorem on preservation of (co)limits by adjoint functors one can often conclude that certain functors cannot have a right or a left adjoint.

Examples

a) The forgetful functor Mon $\to$ Set does not preserve epis, as we have seen in 1.2. In chapter 3 we've seen that f is epi iff is a pushout; since left adjoints preserve identities and pushouts, they preserve epis; therefore the forgetful functor Mon $\to$ Set does not have a right adjoint;

b) The functor $(-) \times X$: Set $\to$ Set does not preserve the terminal object unless X is itself terminal in Set; therefore, it does not have a left adjoint for non-terminal X.

c) The forgetful functor Pos $\to$ Set has a left adjoint, but it cannot have a right adjoint: it preserves all coproducts, including the initial object, but not all coequalizers.

Exercise 106 Prove the last example. Hint: think of the coequalizer of the following two maps $\mathbb{Q} \to \mathbb{R}$: one is the inclusion, the other is the constant zero map.

Another use of the theorem has to do with the computation of limits. Many categories, as we have seen, have a forgetful functor to Set which has a left adjoint. So the forgetful functor preserves limits, and since these can easily be computed in Set, one already knows the "underlying set" of the vertex of the limiting cone one wants to compute.

Does a converse to the theorem hold? I.e. given $G : \mathcal{C} \to \mathcal{D}$ which preserves all limits; does G have a left adjoint? In general *no*, unless $\mathcal{C}$ is sufficiently complete, and a rather technical condition, the "solution set condition" holds. The *adjoint functor theorem* (Freyd) tells that in that case there is a converse:

Definition 5.3 (Solution set condition) *$G : \mathcal{C} \to \mathcal{D}$ satisfies the solution set condition (ssc) for an object D of $\mathcal{D}$, if there is a* set *X_D of objects of $\mathcal{C}$, such that every arrow $D \to GC$ factors as*

$$\begin{tikzcd} D \arrow[r] \arrow[dr] & GC \\ & GC' \arrow[u, "G(f)"'] \end{tikzcd}$$

for some $C' \in X_D$.

Theorem 5.4 (Adjoint Functor Theorem) *Let $\mathcal{C}$ be a locally small, complete category and $G : \mathcal{C} \to \mathcal{D}$ a functor. G has a left adjoint if and only if G preserves all small limits and satisfies the ssc for every object D of $\mathcal{D}$.*

Proof. I sketch the proof for the 'if' part; convince yourself that the 'only if' part is trivial.

For any object D of $\mathcal{D}$ let $D{\downarrow}G$ be the category defined in exercise 96. By that exercise, we are looking for an initial object of $D{\downarrow}G$.

The solution set condition means, that there is a set $\mathcal{K}_0$ of objects of $D{\downarrow}G$ such that for any object x of $D{\downarrow}G$ there is an element $k \in \mathcal{K}_0$ and an arrow $k \to x$ in $D{\downarrow}G$.

The fact that $\mathcal{C}$ is complete and that G preserves all small limits, entails that $D{\downarrow}G$ is complete. Moreover, $D{\downarrow}G$ is locally small as $\mathcal{C}$ is. Now let $\mathcal{K}$ be the full subcategory of $D{\downarrow}G$ with set of objects $\mathcal{K}_0$. Then since $D{\downarrow}G$ is locally small and $\mathcal{K}_0$ a set, $\mathcal{K}$ is small. Take, by completeness of $\mathcal{C}$, a vertex of a limiting cone for the inclusion: $\mathcal{K} \to D{\downarrow}G$. Call this vertex x_0. x_0 may not yet be an initial object of $D{\downarrow}G$, but now let $\mathcal{M}$ be the full subcategory of $D{\downarrow}G$ on the single object x_0 (M is a monoid), and let x be a vertex of a limiting cone for the inclusion $\mathcal{M} \to D{\downarrow}G$. x is the *joint equalizer* of all arrows $f : x_0 \to x_0$ in $D{\downarrow}G$, and this will be an initial object in $D{\downarrow}G$. ∎

Let me remark that in natural situations, the ssc is always satisfied. But then in those situations, one generally does not invoke the Adjoint Functor Theorem in order to conclude to the existence of a left adjoint. The value of this theorem is theoretical, rather than practical.

For small categories $\mathcal{C}$, the ssc is of course irrelevant. But categories which are small *and* complete are complete preorders, as we saw in chapter 3.

For preorders $\mathcal{C}, \mathcal{D}$ we have: if $\mathcal{C}$ is complete, then $G : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if G preserves all limits, that is: greatest lower bounds $\bigwedge B$ for all $B \subseteq \mathcal{C}$. For, put

$$F(d) = \bigwedge\{c | d \leq G(c)\}$$

Then $F(d) \leq c'$ implies (since G preserves $\bigwedge$) $\bigwedge\{G(c)|d \leq G(c)\} \leq G(c')$ which implies $d \leq G(c')$ since $d \leq \bigwedge\{G(c)|d \leq G(c)\}$; conversely, $d \leq G(c')$ implies $c' \in \{c|d \leq G(c)\}$ so $F(d) = \bigwedge\{c|d \leq G(c)\} \leq c'$.

6 Monads and Algebras

Given an adjunction $(F, G, \varepsilon, \eta)$: $\mathcal{C} \leftrightarrows \mathcal{D}$ let us look at the functor $T = GF : \mathcal{D} \to \mathcal{D}$.

We have a natural transformation $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow T$ and a natural transformation $\mu : T^2 \Rightarrow T$. The components μ_D are

$$T^2(D) = GFGFD \overset{G(\varepsilon_{FD})}{\to} GFD = T(D)$$

Furthermore the equalities

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

hold. Here $(T\mu)_D = T(\mu_D) : T^3D \to TD$ and $(\mu T)_D = \mu_{TD} : T^3D \to TD$ (Similar for ηT and $T\eta$).

Exercise 107 Prove these equalities.

A triple (T, μ, η) satisfying these identities is called a *monad*. Try to see the formal analogy between the defining equalities for a monad and the axioms for a monoid: writing $m(e, f)$ for ef in a monoid, and η for the unit element, we have

$$\begin{array}{ll} m(e, m(g, h)) = m(m(e, g), h) & \text{(associativity)} \\ m(\eta, e) = m(e, \eta) = e & \text{(unit)} \end{array}$$

Following this one calls μ the *multiplication* of the monad, and η its *unit*.

Example. The powerset functor $\mathcal{P}$: Set $\to$ Set (example j) of 2.2, with η and μ indicated there) is a monad (check).

Dually, there is the notion of a *comonad* (L, δ, ε) on a category $\mathcal{C}$, with equalities

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"'] & L^3 \end{tikzcd} \quad \begin{tikzcd} & L \arrow[dl, "\mathrm{id}_L"'] \arrow[d, "\delta"] \arrow[dr, "\mathrm{id}_L"] & \\ L & L^2 \arrow[l, "\varepsilon L"] \arrow[r, "L\varepsilon"'] & L \end{tikzcd}$$

Given an adjunction $(F, G, \varepsilon, \eta)$, $(FG, \delta = F\eta G, \varepsilon)$ is a comonad on $\mathcal{C}$. We call δ the *comultiplication* and ε the *counit* (this is in harmony with the unit-counit terminology for adjunctions).

Although, in many contexts, comonads and the notions derived from them are at least as important as monads, the treatment is dual so I concentrate on monads.

Every adjunction gives rise to a monad; conversely, every monad arises from an adjunction, but in more than one way. Essentially, there are a maximal and a minimal solution to the problem of finding an adjunction from which a given monad arises.

Example. A monad on a poset P is a monotone function $T : P \to P$ with the properties $x \le T(x)$ and $T^2(x) \le T(x)$ for all $x \in P$; such an operation is also often called a *closure operation* on P. Note that $T^2 = T$ because T is monotone.

In this situation, let $Q \subseteq P$ be the image of T, with the ordering inherited from P. We have maps $r : P \to Q$ and $i : Q \to P$ such that ri is the identity on Q and $ir = T : P \to P$.

For $x \in P$, $y \in Q$ we have $x \le i(y) \Leftrightarrow r(x) \le y$ (check); so $r \dashv i$ and the operation T arises from this adjunction.

6.1 Algebras for a monad

Given a monad (T, η, μ) on a category $\mathcal{C}$, we define the category T-Alg of *T-algebras* as follows:

- *Objects* are pairs (X, h) where X is an object of $\mathcal{C}$ and $h : T(X) \to X$ is an arrow in $\mathcal{C}$ such that

$$\begin{tikzcd} T^2(X) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\ T(X) \arrow[r, "h"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & T(X) \arrow[d, "h"] \\ & X \end{tikzcd}$$

  commute;

- *Morphisms*: $(X, h) \to (Y, k)$ are morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ for which

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "h"'] & T(Y) \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  commutes.

Theorem 6.1 *There is an adjunction between T-Alg and $\mathcal{C}$ which brings about the given monad T.*

Proof. There is an obvious forgetful functor $U^T : T\text{-Alg} \to \mathcal{C}$ which takes (X, h) to X. I claim that U^T has a left adjoint F^T:

F^T assigns to an object X the T-algebra $T^2(X) \xrightarrow{\mu_X} T(X)$; to $X \xrightarrow{f} Y$ the map $T(f)$; this is an algebra map because of the naturality of μ. That $T^2(X) \xrightarrow{\mu_X} T(X)$ is an algebra follows from the defining axioms for a monad T.

Now given any arrow $g : X \to U^T(Y,h)$ we let $\tilde{g} : (T(X), \mu_X) \to (Y,h)$ be the arrow $T(X) \stackrel{T(g)}{\to} T(Y) \stackrel{h}{\to} Y$. This is a map of algebras since

$$\begin{tikzcd}
T^2(X) \arrow[r, "T^2(g)"] \arrow[d, "\mu_X"'] & T^2(Y) \arrow[r, "T(h)"] \arrow[d, "\mu_Y"'] & T(Y) \arrow[d, "h"] \\
T(X) \arrow[r, "T(g)"'] & T(Y) \arrow[r, "h"'] & Y
\end{tikzcd}$$

commutes; the left hand square is the naturality of μ; the right hand square is because (Y,h) is a T-algebra.

Conversely, given a map of algebras $f : (TX, \mu_X) \to (Y,h)$ we have an arrow $\bar{f} : X \to Y$ by taking the composite $X \stackrel{\eta_X}{\to} TX \stackrel{f}{\to} Y$.

Now $\tilde{\bar{f}} : TX \to Y$ is the composite

$$TX \stackrel{T(\eta_X)}{\to} T^2X \stackrel{T(f)}{\to} TY \stackrel{h}{\to} Y$$

Since f is a T-algebra map, this is

$$T(X) \stackrel{T(\eta_X)}{\to} T^2(X) \stackrel{\mu_X}{\to} T(X) \stackrel{f}{\to} Y$$

which is f by the monad laws.

Conversely, $\bar{\tilde{g}} : X \to Y$ is the composite

$$X \stackrel{\eta_X}{\to} TX \stackrel{T(g)}{\to} TY \stackrel{h}{\to} Y$$

By naturality of η and the fact that (Y,h) is a T-algebra, we conclude that $\bar{\tilde{g}} = g$. So we have a natural 1-1 correspondence

$$\mathcal{C}(X, U^T(Y,h)) \simeq T\text{-Alg}(F^T(X), (Y,h))$$

and our adjunction.

Note that the composite U^TF^T is the functor T, and that the unit η of the adjunction is the unit of T; the proof that for the counit ε of $F^T \dashv U^T$ we have that

$$T^2 = U^TF^TU^TF^T \stackrel{U^T\varepsilon F^T}{\to} U^TF^T = T$$

is the original multiplication μ, is left to you. ∎

Exercise 108 Complete the proof.

Example. The *group monad*. Combining the forgetful functor $U : \mathrm{Grp} \to \mathrm{Set}$ with the left adjoint, the free functor $\mathrm{Set} \to \mathrm{Grp}$, we get the following monad on Set:

$T(A)$ is the set of sequences on the alphabet $A \sqcup A^{-1}$ (A^{-1} is the set $\{a^{-1} | a \in A\}$ of formal inverses of elements of A, as in example e) of 1.1) in which no

aa^{-1} or $a^{-1}a$ occur. The unit $A \stackrel{\eta_A}{\to} TA$ sends $a \in A$ to the string $\langle a \rangle$. The multiplication $\mu : T^2(A) \to T(A)$ works as follows. Define $(-)^- : A \sqcup A^{-1} \to A \sqcup A^{-1}$ by $a^- = a^{-1}$ and $(a^{-1})^- = a$. Define also $(-)^-$ on strings by $(a_1 \dots a_n)^- = a_n^- \dots a_1^-$. Now for an element of $TT(A)$, which is a string on the alphabet $T(A) \sqcup T(A)^{-1}$, say $\sigma_1 \dots \sigma_n$, we let $\mu_A(\sigma_1 \dots \sigma_n)$ be the concatenation of the strings $\tilde{\sigma}_1, \dots, \tilde{\sigma}_n$ on the alphabet $A \sqcup A^{-1}$, where $\tilde{\sigma}_i = \sigma_i$ if $\sigma_i \in T(A)$, and $\tilde{\sigma}_i = (\sigma_i)^-$ if $\sigma_i \in T(A)^{-1}$. Of course we still have to remove possible substrings of the form aa^{-1} etc.

Now let us look at algebras for the group monad: maps $T(A) \stackrel{h}{\to} A$ such that for a string of strings

$$\alpha = \sigma_1, \dots, \sigma_n = \langle \langle s_1^1, \dots, s_1^{k_1} \rangle, \dots, \langle s_n^1, \dots, s_n^{k_n} \rangle \rangle$$

we have that

$$h(\langle h\sigma_1, \dots, h\sigma_n \rangle) = h(\langle s_1^1, \dots, s_1^{k_1}, \dots, s_n^1, \dots, s_n^{k_n} \rangle)$$

and

$$h(\langle a \rangle) = a \text{ for } a \in A$$

I claim that this is the same thing as a group structure on A, with multiplication $a \cdot b = h(\langle a, b \rangle)$.

The unit element is given by $h(\langle\rangle)$; the inverse of $a \in A$ is $h(\langle a^{-1} \rangle)$ since

$$\begin{array}{rcl} h(\langle a, h(\langle a^{-1} \rangle) \rangle) & = & h(\langle h(\langle a \rangle), h(\langle a^{-1} \rangle) \rangle) = \\ h(\langle a, a^{-1} \rangle) & = & h(\langle\rangle), \text{ the unit element} \end{array}$$

Try to see for yourself how the associativity of the monad and its algebras transforms into associativity of the group law.

Exercise 109 Finish the proof of the theorem: for the group monad T, there is an equivalence of categories between T-Alg and Grp.

This situation is very important and has its own name:

Definition 6.2 *Given an adjunction* $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, $F \dashv G$, *there is always a* comparison functor $K : \mathcal{C} \to T$-*Alg for* $T = GF$, *the monad induced by the adjunction. K sends an object C of* $\mathcal{C}$ *to the T-algebra* $GFG(C) \stackrel{G(\varepsilon_C)}{\to} G(C)$.

We say that the functor $G : \mathcal{C} \to \mathcal{D}$ *is* monadic, *or by abuse of language (if G is understood), that* $\mathcal{C}$ is monadic over $\mathcal{D}$, *if K is an equivalence.*

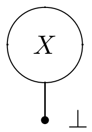
Exercise 110 Check that $K(C)$ is a T-algebra. Complete the definition of K as a functor. Check that in the example of the group monad, the functor T-Alg $\to$ Grp defined there is a pseudo inverse to the comparison functor K.

In many cases however, the situation is not monadic. Take the forgetful functor U : Pos $\to$ Set. It has a left adjoint F which sends a set X to the discrete ordering on X ($x \leq y$ iff $x = y$). Of course, UF is the identity on Set and the UF-algebras are just sets. The comparison functor K is the functor U, and this is not an equivalence.

Exercise 111 Why not?

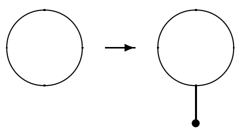
Another example of a monadic situation is of importance in domain theory. Let $\mathrm{Pos}_\perp$ be the category of partially ordered sets with a least element, and order preserving maps which also preserve the least element.

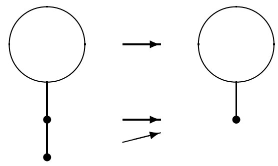
There is an obvious inclusion functor $U : \mathrm{Pos}_\perp \to \mathrm{Pos}$, and U has a left adjoint F. Given a poset X, $F(X)$ is X "with a bottom element added":



Given $f : X \to Y$ in Pos, $F(f)$ sends the new bottom element of X to the new bottom element of Y, and is just f on the rest. If $f : X \to Y$ in Pos is a map and Y has a least element, we have $F(X) \to Y$ in $\mathrm{Pos}_\perp$ by sending $\perp$ to the least element of Y. So the adjunction is clear.

The monad $UF : \mathrm{Pos} \to \mathrm{Pos}$, just adding a least element, is called the *lifting monad*. Unit and multiplication are:





$\eta_X : X \to T(X)$ $\qquad\qquad$ $\mu_X : T^2(X) \to T(X)$

A T-algebra $h : TX \to X$ is first of all a monotone map, but since $h\eta_X = \mathrm{id}_X$, h is epi in Pos so surjective. It follows that X must have a least element $h(\perp)$. From the axioms for an algebra one deduces that h must be the identity when restricted to X, and $h(\perp)$ the least element of X.

Exercise 112 Given $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, $F \dashv G$, $T = GF$. Prove that the comparison functor $K : \mathcal{C} \to T\text{-Alg}$ satisfies $U^T K = G$ and $KF = F^T$ where $T\text{-Alg} \underset{U^T}{\overset{F^T}{\leftrightarrows}} \mathcal{D}$ as in theorem 6.1.

Another poset example: algebras for the power set monad $\mathcal{P}$ on Set (example j) of 2.2). Such an algebra $h : \mathcal{P}(X) \to X$ must satisfy $h(\{x\}) = x$ and for $\alpha \subseteq \mathcal{P}(X)$:

$$h(\{h(A) | A \in \alpha\}) = h(\{x | \exists A \in \alpha(x \in A)\}) = h(\bigcup \alpha)$$

Now we can, given an algebra structure on X, define a partial order on X by putting $x \le y$ iff $h(\{x, y\}) = y$.

Indeed, this is clearly reflexive and antisymmetric. As to transitivity, if $x \le y$ and $y \le z$ then

$$\begin{array}{rl} h(\{x,z\}) = & h(\{x, h(\{y,z\})\}) \quad = \\ h(\{h(\{x\}), h(\{y,z\})\}) = & h(\{x\} \cup \{y,z\}) \quad = \\ h(\{x,y\} \cup \{z\}) = & h(\{h(\{x,y\}), h(\{z\})\}) = \\ h(\{y,z\}) = & z \end{array}$$

so $x \le z$.

Furthermore this partial order is *complete*: least upper bounds for arbitrary subsets exist. For $\bigvee B = h(B)$ for $B \subseteq X$: for $x \in B$ we have $h(\{x, h(B)\}) = h(\{x\} \cup B) = h(B)$ so $x \le \bigvee B$; and if $x \le y$ for all $x \in B$ then

$$\begin{array}{rl} h(\{h(B), y\}) = & h(B \cup \{y\}) \quad = \\ h(\bigcup_{x \in B} \{x, y\}) = & h(\{h(\{x,y\}) | x \in B\}) = \\ h(\{y\}) = & y \end{array}$$

so $\bigvee B \le y$.

We can also check that a map of algebras is a $\bigvee$-preserving monotone function. Conversely, every $\bigvee$-preserving monotone function between complete posets determines a $\mathcal{P}$-algebra homomorphism.

We have an equivalence between the category of complete posets and $\bigvee$-preserving functions, and $\mathcal{P}$-algebras.

Exercise 113 Let $P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ be the contravariant powerset functor, and $\bar{P}$ its left adjoint, as in j) of 5.1. Let $T : \mathrm{Set} \to \mathrm{Set}$ the induced monad.

a) Describe unit and multiplication of this monad explicitly.

b) Show that $\mathrm{Set}^{\mathrm{op}}$ is equivalent to T-Alg [Hint: if this proves hard, have a look at VI.4.3 of Johnstone's "Stone Spaces"].

c) Conclude that there is an adjunction

$$\mathrm{CABool} \rightleftarrows \mathrm{Set}$$

which presents CABool as monadic over Set.

6.2 T-Algebras at least as complete as $\mathcal{D}$

Let T be a monad on $\mathcal{D}$. The following exercise is meant to show that if $\mathcal{D}$ has all limits of a certain type, so does T-Alg. In particular, if $\mathcal{D}$ is complete, so is T-Alg; this is often an important application of a monadic situation.

Exercise 114 Let $\mathcal{E}$ be a category such that every functor $M : \mathcal{E} \to \mathcal{D}$ has a limiting cone. Now suppose $M : \mathcal{E} \to T$-Alg. For objects E of $\mathcal{E}$, let $M(E)$ be the T-algebra $T(m_E) \stackrel{h_E}{\to} m_E$.

a) Let $(D, (\nu_E | E \in \mathcal{E}_0))$ be a limiting cone for $U^T M : \mathcal{E} \to \mathcal{D}$. Using the T-algebra structure on $M(E)$ and the fact that $U^T M(E) = m_E$, show that there is also a cone $(TD, (\pi_E | E \in \mathcal{E}_0))$ for $U^T M$;

b) Show that the unique map of cones: $(TD, \pi) \to (D, \nu)$ induces a T-algebra structure $TD \xrightarrow{h} D$ on D;

c) Show that $TD \xrightarrow{h} D$ is the vertex of a limiting cone for M in T-Alg.

This exercise gives a situation which has its own name. For a functor $G : \mathcal{C} \to \mathcal{D}$ we say that G *creates limits* of type $\mathcal{E}$ if for every functor $M : \mathcal{E} \to \mathcal{C}$ and every limiting cone (D, μ) for GM in $\mathcal{D}$, there is a unique cone (C, ν) for M in $\mathcal{C}$ which is taken by G to (D, μ), and moreover this unique cone is limiting for M in $\mathcal{C}$.

Clearly, if G creates limits of type $\mathcal{E}$ and $\mathcal{D}$ has all limits of type $\mathcal{E}$, then $\mathcal{C}$ has them, too. The exercise proves that the forgetful functor $U^T : T - \mathrm{Alg} \to \mathcal{D}$ creates limits of every type.

6.3 The Kleisli category of a monad

I said before that for a monad T on a category $\mathcal{D}$, there are a "maximal and a minimal solution" to the problem of finding an adjunction which induces the given monad.

We've seen the category T-Alg, which we now write as $\mathcal{D}^T$; we also write $G^T : T\text{-Alg} \to \mathcal{D}$ for the forgetful functor. In case T arises from an adjunction $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, there was a comparison functor $\mathcal{C} \xrightarrow{K} \mathcal{D}^T$. In the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "K"] \arrow[dr, shift left, "G"'] & & \mathcal{D}^T \arrow[dl, shift left, "G^T"] \\ & \mathcal{D} \arrow[ul, shift left, "F"'] \arrow[ur, shift left, "F^T"'] & \end{tikzcd}$$

we have that $KF = F^T$ and $G^T K = G$.

Moreover, the functor K is unique with this property.

This leads us to define a category T-Adj of adjunctions $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ such that $GF = T$. A *map* of such T-adjunctions from $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ to $\mathcal{C}' \underset{G'}{\overset{F'}{\leftrightarrows}} \mathcal{D}$ is a functor $K : \mathcal{C} \to \mathcal{C}'$ satisfying $KF = F'$ and $G'K = G$.

What we have proved about T-Alg can be summarized by saying that the adjunction $\mathcal{D}^T \underset{G^T}{\overset{F^T}{\leftrightarrows}} \mathcal{D}$ is a *terminal object* in T-Adj. This was the "maximal" solution.

T-Adj has also an initial object: the *Kleisli category* of T, called $\mathcal{D}_T$. $\mathcal{D}_T$ has the same objects as $\mathcal{D}$, but a map in $\mathcal{D}_T$ from X to Y is an arrow $X \xrightarrow{f} T(Y)$

in $\mathcal{D}$. Composition is defined as follows: given $X \xrightarrow{f} T(Y)$ and $Y \xrightarrow{g} T(Z)$ in $\mathcal{D}$, considered as a composable pair of morphisms in $\mathcal{D}_T$, the composition gf in $\mathcal{D}_T$ is the composite

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T^2(Z) \xrightarrow{\mu_Z} T(Z)$$

in $\mathcal{D}$.

Exercise 115 Prove that composition is associative. What are the identities of $\mathcal{D}_T$?

The adjunction $\mathcal{D}_T \underset{G_T}{\overset{F_T}{\leftrightarrows}} \mathcal{D}$ is defined as follows: the functor G_T sends the object X to $T(X)$ and $f : X \to Y$ ($f : X \to T(Y)$ in $\mathcal{D}$) to

$$T(X) \xrightarrow{T(f)} T^2(Y) \xrightarrow{\mu_Y} T(Y)$$

The functor F_T is the identity on objects and sends $X \xrightarrow{f} Y$ to $X \xrightarrow{f} Y \xrightarrow{\eta_Y} T(Y)$, considered as $X \to Y$ in $\mathcal{D}_T$.

Exercise 116 Define unit and counit; check $F_T \dashv G_T$.

Exercise 117 Let T be a monad on $\mathcal{D}$. Call an object of T-Alg *free* if it is in the image of $F^T : \mathcal{D} \to T-\text{Alg}$. Show that the Kleisli category $\mathcal{D}_T$ is equivalent to the full subcategory of T-Alg on the free T-algebras.

Now for every adjunction $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ with $GF = T$, there is a unique comparison functor $L : \mathcal{D}_T \to \mathcal{C}$ such that $GL = G_T$ and $LF_T = F$.

L sends the object X to $F(X)$ and $f : X \to Y$ (so $f : X \to T(Y) = GF(Y)$ in $\mathcal{D}$) to its transpose $\tilde{f} : F(X) \to F(Y)$.

Exercise 118 Check the commutations. Prove the uniqueness of L w.r.t. these properties.

Exercise 119 Let Rng1 be the category of rings with unit and unitary ring homomorphisms. Since every ring with 1 is a (multiplicative) monoid, there is a forgetful functor $G : \text{Rng1} \to \text{Mon}$. For a monoid M, let $Z[M]$ be the ring of formal expressions

$$n_1c_1 + \cdots + n_kc_k$$

with $k \geq 0$, $n_1, \ldots, n_k \in Z$ and $c_1, \ldots, c_k \in M$. This is like a ring of polynomials, but multiplication uses the multiplication in M. Show that this defines a functor $F : \text{Mon} \to \text{Rng1}$ which is left adjoint to G, and that G is monadic, i.e. the category of GF-algebras is equivalent to Rng1. [Hint: Proceed as in the example of the powerset monad. That is, let $h : GF(M) \to M$ be a monoid homomorphism which gives M the structure of a GF-algebra. Find an abelian group structure on M such that M becomes a ring with unit]

Exercise 120 What does the Kleisli category for the covariant powerset monad look like?

7 Cartesian closed categories and the λ-calculus

Many set-theoretical constructions are completely determined (up to isomorphism, as always) by their categorical properties in Set. We are therefore tempted to generalize them to arbitrary categories, by taking the characteristic categorical property as a definition. Of course, this procedure is not really well-defined and it requires sometimes a real insight to pick the 'right' categorical generalization. For example, the category of sets has very special properties:

- $f : X \to Y$ is mono if and only if $fg = fh$ implies $g = h$ for any two maps $g, h : 1 \to X$, where 1 is a terminal object (we say 1 is *a generator*);
- objects X and Y are isomorphic if there exist monos $f : X \to Y$ and $g : Y \to X$ (the Cantor-Bernstein theorem);
- every mono $X \xrightarrow{f} Y$ is part of a coproduct diagram

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

And if you believe the axiom of choice, there is its categorical version:

- Every epi is split

None of these properties is generally valid, and categorical generalizations based on them are usually of limited value.

In this chapter we focus on a categorical generalization of a set-theoretical concept which has proved to have numerous applications: Cartesian closed categories as the generalization of "function space".

In example f) of 5.1 we saw that the set of functions Z^X appears as the value at Z of the right adjoint to the product functor $(-) \times X$. A category is called *cartesian closed* if such right adjoints always exist. In such categories we may really think of this right adjoint as giving the "object of functions (or arrows)", as the treatment of the λ-calculus will make clear.

7.1 Cartesian closed categories (ccc's); examples and basic facts

Definition 7.1 *A category $\mathcal{C}$ is called* cartesian closed *or* a ccc *if it has finite products, and for every object X of $\mathcal{C}$ the product functor $(-) \times X$ has a right adjoint.*

Of course, "the" product functor only exists once we have chosen a product diagram for every pair of objects of $\mathcal{C}$. In this chapter we assume that we have such a choice, as well as a distinguished terminal object 1; and we assume also that for each object X we have a *specified* right adjoint to the functor $(-) \times X$, which we write as $(-)^X$ (Many authors write $X \Rightarrow (-)$, but I think that overloads the arrows notation too much). Objects of the form Z^X are called *exponents*.

We have the *unit*

$$Y \stackrel{\eta_{Y,X}}{\to} (Y \times X)^X$$

and *counit*

$$Y^X \times X \stackrel{\varepsilon_{Y,X}}{\to} Y$$

of the adjunction $(-) \times X \dashv (-)^X$. Anticipating the view of Y^X as the object of arrows $X \to Y$, we call ε *evaluation*.

Examples

a) A preorder (or partial order) is cartesian closed if it has a top element 1, binary meets $x \wedge y$ and for any two elements x, y an element $x \to y$ satisfying for each z:

$$z \leq x \to y \text{ iff } z \wedge x \leq y$$

b) Set is cartesian closed; Cat is cartesian closed (2.1);

c) Top is not cartesian closed. In chapter 4 it was remarked, that for non-locally compact spaces X, the functor $X \times (-)$ will not preserve quotients (coequalizers); hence, it cannot have a right adjoint. There are various subcategories of Top which are cartesian closed, if one takes as exponent Y^X the set of continuous maps $Y \to X$, topologized with the compact-open topology.

d) Pos is cartesian closed. The exponent Y^X is the set of all monotone maps $X \to Y$, ordered pointwise ($f \leq g$ iff for all $x \in X$, $fx \leq gx$ in Y);

e) Grp and Abgp are not cartesian closed. In both categories, the initial object is the one-element group. Since for non-initial groups G, $(-) \times G$ does not preserve the initial object, it cannot have a right adjoint (the same argument holds for Mon);

f) **1** is cartesian closed; **0** isn't (why?);

g) $\text{Set}^{\mathcal{C}^{\text{op}}}$ is cartesian closed. Products and 1 are given "pointwise" (in fact all limits are), that is $F \times G(C) = F(C) \times G(C)$ and $1(C)$ is the terminal 1 in Set, for all $C \in \mathcal{C}_0$.

The construction of the exponent G^F is a nice application of the Yoneda lemma. Indeed, for G^F to be the right adjoint (at G) of $(-) \times F$, we need for every object C of $\mathcal{C}$:

$$\text{Set}^{\mathcal{C}^{\text{op}}}(h_C \times F, G) \simeq \text{Set}^{\mathcal{C}^{\text{op}}}(h_C, G^F) \simeq G^F(C)$$

where the last isomorphism is by the Yoneda lemma.

Now the assignment $C \mapsto \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C \times F, G)$ defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, which we denote by G^F. At the same time, this construction defines a functor $(-)^F : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, which is right adjoint to $(-) \times F$. It is a nice exercise to prove this.

h) A monoid is never cartesian closed unless it is trivial. However, if from the definition of 'cartesian closed' one would delete the requirement that it has a terminal object, an interesting class of 'cartesian closed' monoids exists: the *C-mnoids* in the book "Higher Order Categorical Logic" by J. Lambek and Ph. Scott.

Exercise 121 Show that every Boolean algebra is cartesian closed (as a partial order).

Exercise 122 Show that CABool is not cartesian closed [use 2.3].

Exercise 123 Show that a complete partial order is cartesian closed if and only if it's a frame [see section 4.5].

Exercise 124 Let Ω be a frame. By the preceding exercise, it is cartesian closed; denote by $x{\to}y$ the exponent in Ω. This exercise is meant to let you show that the category $\mathcal{C}_\Omega$ from section 4.5 is cartesian closed.

a) Show that Ω also has greatest lower bounds $\bigwedge B$ for all subsets B.

b) Given objects (X, E_X) and (Y, E_Y), define their exponent $(Y, E_Y)^{(X,E_X)}$ as (Y^X, E) where Y^X is the set of all functions $X \to Y$ in Set, and

$$E(f) = \bigwedge\{E_X(x){\to}E_Y(f(x)) | x \in X\}$$

Show that this defines a right adjoint (at (Y, E_Y)) of $(-) \times (X, E_X)$.

Some useful facts:

- $\mathcal{C}$ is cartesian closed if and only if it has finite products, and for each pair of objects X, Y there is an object Y^X and an arrow $\varepsilon : Y^X \times X \to Y$ such that for every Z and map $Z \times X \xrightarrow{f} Y$ there is a unique $Z \xrightarrow{\tilde{f}} Y^X$ such that

$$\begin{tikzcd} Z \times X \arrow[rr, "f"] \arrow[dr, "\tilde{f} \times \mathrm{id}_X"'] & & Y \\ & Y^X \times X \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

  commutes (use the result of exercise 96).

- In a ccc, there are natural isomorphisms $1^X \simeq 1$; $(Y \times Z)^X \simeq Y^X \times Z^X$; $(Y^Z)^X \simeq Y^{Z \times X}$.

- If a ccc has coproducts, we have $X \times (Y+Z) \simeq (X \times Y) + (X \times Z)$ and $Y^{Z+X} \simeq Y^Z \times Y^X$.

Exercise 125 Prove these facts.

Recall that two maps $Z \times X \to Y$ and $Z \to Y^X$ which correspond to each other under the adjunction are called each other's *transposes*.

Exercise 126 In a ccc, prove that the transpose of a composite $Z \xrightarrow{g} W \xrightarrow{f} Y^X$ is

$$Z \times X \xrightarrow{g \times \mathrm{id}_X} W \times X \xrightarrow{\tilde{f}} Y$$

if $\tilde{f}$ is the transpose of f.

Lemma 7.2 *In a ccc, given $f : X' \to X$ let $Y^f : Y^X \to Y^{X'}$ be the transpose of*

$$Y^X \times X' \xrightarrow{\mathrm{id} \times f} Y^X \times X \xrightarrow{\varepsilon} Y$$

Then for each $f : X' \to X$ and $g : Y \to Y'$ the diagram

$$\begin{tikzcd}
Y^X \arrow[r, "g^X"] \arrow[d, "Y^f"'] & Y'^X \arrow[d, "Y'^f"] \\
Y^{X'} \arrow[r, "g^{X'}"'] & Y'^{X'}
\end{tikzcd}$$

commutes.

Proof. By the exercise, the transposes of both composites are the top and bottom composites of the following diagram:

$$\begin{tikzcd}
 & Y'^X \times X' \arrow[r, "\mathrm{id} \times f"] & Y'^X \times X \arrow[dr, "\varepsilon"] & \\
Y^X \times X' \arrow[ur, "g^X \times \mathrm{id}"] \arrow[r, "\mathrm{id} \times f"] \arrow[dr, "Y^f \times \mathrm{id}"'] & Y^X \times X \arrow[ur, "g^X \times \mathrm{id}"] \arrow[r, "\varepsilon"] & Y \arrow[r, "g"] & Y' \\
 & Y^{X'} \times X' \arrow[ur, "\varepsilon"] \arrow[r, "g^{X'} \times \mathrm{id}"'] & Y'^{X'} \times X' \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

This diagram commutes because the right hand "squares" are naturality squares for ε, the lower left hand square commutes because both composites are the transpose of Y^f, and the upper left hand square commutes because both composites are $g^X \times f$. ∎

Proposition 7.3 *For every ccc $\mathcal{C}$ there is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, assigning Y^X to (X, Y), and given $g : Y \to Y'$ and $f : X' \to X$, $g^f : Y^X \to Y'^{X'}$ is either of the composites in the lemma.*

Exercise 127 Prove the proposition.

7.2 Typed λ-calculus and cartesian closed categories

The λ-calculus is an extremely primitive formalism about functions. Basically, we can form functions (by λ-abstraction) and apply them to arguments; that's all. Here I treat briefly the *typed* λ-calculus.

We start with a set $\mathcal{S}$ of *type symbols* $S_1, S_2, \ldots$

Out of $\mathcal{S}$ we make the set of types as follows: every type symbol is a type, and if T_1 and T_2 are types then so is $(T_1{\Rightarrow}T_2)$.

We have also *terms* of each type (we label the terms like $t{:}T$ to indicate that t is a term of type T):

- we may have constants $c{:}T$ of type T;
- for every type T we have a denumerable set of variables $x_1{:}T, x_2{:}T, \ldots$;
- given a term $t{:}(T_1{\Rightarrow}T_2)$ and a term $s{:}T_1$, there is a term $(ts){:}T_2$;
- given $t{:}T_2$ and a variable $x{:}T_1$ there is a term $\lambda x.t{:}T_1{\Rightarrow}T_2$.

Terms $\lambda x.t$ are said to be formed by λ-*abstraction.* This procedure binds the variable x in t. Furthermore we have the usual notion of substitution for free variables in a term t (types have to match, of course). Terms of form (ts) are said to be formed by *application.*

In the λ-calculus, the only statements we can make are equality statements about terms. Again, I formulate the rules in terms of theories. First, let us say that a *language* consists of a set of type symbols and a set of constants, each of a type generated by the set of type symbols.

An *equality judgement* is an expression of the form $\Gamma|t = s{:}T$ (to be read: "Γ thinks that s and t are equal terms of type T "), where Γ is a finite set of variables which includes all the variables free in either t or s, and t and s are terms of type T.

A *theory* is then a set $\mathcal{T}$ of equality judgements which is closed under the following rules:

i) $\Gamma|t = s{:}T$ in $\mathcal{T}$ implies $\Delta|t = s{:}T$ in $\mathcal{T}$ for every superset Δ of Γ;

ii) $FV(t)|t = t{:}T$ is in $\mathcal{T}$ for every term $t{:}T$ of the language (again, $FV(t)$ is the set of free variables of t);
if $\Gamma|t = s{:}T$ and $\Gamma|s = u{:}T$ are in $\mathcal{T}$ then so is $\Gamma|t = u{:}T$;

iii) if $t(x_1, \ldots, x_n){:}T$ is a term of the language, with free variables $x_1{:}S_1,\ldots,x_n{:}S_n$, and $\Gamma|s_1 = t_1{:}S_1, \ldots, \Gamma|s_n = t_n{:}S_n$ are in $\mathcal{T}$ then

$$\Gamma|t[s_1/x_1, \ldots, s_n/x_n] = t[t_1/x_1, \ldots, t_n/x_n]{:}T$$

is in $\mathcal{T}$;

iv) if t and s are terms of type $(T_1{\Rightarrow}T_2)$, x a variable of type T_1 which does not occur in t or s, and $\Gamma \cup \{x\}|(tx) = (sx){:}T_2$ is in $\mathcal{T}$, then $\Gamma \setminus \{x\}|t = s{:}(T_1{\Rightarrow}T_2)$ is in $\mathcal{T}$;

v) if $s{:}T_1$ and $t{:}T_2$ are terms and x a variable of type T_2, then

$$FV(s) \setminus \{x\} \cup FV(t) | ((\lambda x.s)t) = s[t/x]{:}T_1$$

is in $\mathcal{T}$.

Given a language, an interpretation of it into a ccc $\mathcal{C}$ starts by choosing objects $[\![S]\!]$ of $\mathcal{C}$ for every type symbol S. This then generates objects $[\![T]\!]$ for every type T by the clause

$$[\![T_1 {\Rightarrow} T_2]\!] = [\![T_2]\!]^{[\![T_1]\!]}$$

The interpretation is completed by choosing interpretations

$$1 \xrightarrow{[\![c]\!]} [\![T]\!]$$

for every constant $c{:}T$ of the language.

Such an interpretation then generates, in much the same way as in chapter 4, interpretations of all terms. For a finite set $\Gamma = \{x_1{:}T_1, \ldots, x_n{:}T_n\}$ let's again write $[\![\Gamma]\!]$ for the product $[\![T_1]\!] \times \cdots \times [\![T_n]\!]$ (this is only defined modulo a permutation of the factors of the product, but that will cause us no trouble).

The interpretation of $t{:}T$ will now be an arrow

$$[\![FV(t)]\!] \xrightarrow{[\![t]\!]} [\![T]\!]$$

defined as follows:

- $[\![x]\!]$ is the identity on $[\![T]\!]$ for every variable $x{:}T$;
- given $[\![t]\!] : [\![FV(t)]\!] \to [\![T_2]\!]^{[\![T_1]\!]}$ and $[\![s]\!] : [\![FV(s)]\!] \to [\![T_1]\!]$ we let $[\![(ts)]\!] : [\![FV((ts))]\!] \to [\![T_2]\!]$ be the composite

$$[\![FV((ts))]\!] \xrightarrow{\langle [\![t]\!]\pi_t, [\![s]\!]\pi_s \rangle} [\![T_2]\!]^{[\![T_1]\!]} \times [\![T_1]\!] \xrightarrow{\varepsilon} [\![T_2]\!]$$

  where π_t and π_s are the projections from $[\![FV((ts))]\!]$ to $[\![FV(t)]\!]$ and $[\![FV(s)]\!]$, respectively;
- given $[\![t]\!] : [\![FV(t)]\!] \to [\![T_2]\!]$ and the variable $x{:}T_1$ we let $[\![\lambda x.t]\!] : [\![FV(t) \setminus \{x\}]\!] \to [\![T_2]\!]^{[\![T_1]\!]}$ be the transpose of

$$[\![FV(t) \setminus \{x\}]\!] \times [\![T_1]\!] \xrightarrow{\tilde{t}} [\![T_2]\!]$$

  where, if x occurs free in t so $[\![FV(t) \setminus \{x\}]\!] \times [\![T_1]\!] \simeq [\![FV(t)]\!]$, $\tilde{t}$ is just $[\![t]\!]$; and if x doesn't occur in t, $\tilde{t}$ is $[\![t]\!]$ composed with the obvious projection.

We now say that an equality judgement $\Gamma|t = s{:}T$ is *true* in this interpretation, if the diagram

$$\begin{tikzcd} & \llbracket FV(t) \rrbracket \arrow[dr, "\llbracket t \rrbracket"] & \\ \llbracket \Gamma \rrbracket \arrow[ur, "\pi_t"] \arrow[dr, "\pi_s"'] & & \llbracket T \rrbracket \\ & \llbracket FV(s) \rrbracket \arrow[ur, "\llbracket s \rrbracket"'] & \end{tikzcd}$$

commutes (again, π_s and π_t projections).

Lemma 7.4 *Let $t(x_1, \ldots, x_n){:}T$ have free variables $x_i{:}T_i$ and let $t_i{:}T_i$ be terms. Let*

$$\tilde{t_i} : \llbracket FV(t[t_1/x_1, \ldots, t_n/x_n]) \rrbracket \to \llbracket T_i \rrbracket$$

be the obvious composite of projection and $\llbracket t_i \rrbracket$.

Then the composition

$$\llbracket FV(t[t_1/x_1, \ldots, t_n/x_n]) \rrbracket \stackrel{\langle \tilde{t_i} | i=1 \ldots n \rangle}{\longrightarrow} \prod_{i=1}^{n} \llbracket T_i \rrbracket = \llbracket FV(t) \rrbracket \stackrel{\llbracket t \rrbracket}{\to} \llbracket T \rrbracket$$

is the interpretation $\llbracket t[t_1/x_1, \ldots, t_n/x_n] \rrbracket$.

Exercise 128 Prove the lemma [take your time. This is not immediate].

Theorem 7.5 *Let S be a set of equality judgements and $\mathcal{T} = Cn(S)$ be the least theory containing S. If every judgement of S is true in the interpretation, so is every judgement in $\mathcal{T}$.*

Proof. Again, we show that the set of true judgements is a theory, i.e. closed under the rules in the definition of a theory.
i) and ii) are trivial;
iii) follows at once by lemma 7.4;
iv) Since $\Gamma \cup \{x\} = (\Gamma \setminus \{x\}) \cup \{x\}$ and because of the inductive hypothesis, we have that

$$\begin{tikzcd} & \llbracket FV(s) \rrbracket \times \llbracket T_1 \rrbracket \arrow[r, "\llbracket s \rrbracket \times \mathrm{id}"'] & \llbracket T_1 \Rightarrow T_2 \rrbracket \times \llbracket T_1 \rrbracket \arrow[dr, "\varepsilon"] & \\ \llbracket \Gamma \setminus \{x\} \cup \{x\} \rrbracket \arrow[ur, "\pi_s \times \mathrm{id}"] \arrow[dr, "\pi_t \times \mathrm{id}"'] & & & \llbracket T_2 \rrbracket \\ & \llbracket FV(t) \rrbracket \times \llbracket T_1 \rrbracket \arrow[r, "\llbracket t \rrbracket \times \mathrm{id}"'] & \llbracket T_1 \Rightarrow T_2 \rrbracket \times \llbracket T_1 \rrbracket \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

commutes. Taking the transposes of both maps, we get the equality we want.
v) According to lemma 7.4, $[\![FV(s[t/x])]\!] \stackrel{[\![s[t/x]]\!]}{\longrightarrow} [\![T_1]\!]$ is

$$[\![FV(s[t/x])]\!] \stackrel{\tilde{t}}{\to} [\![FV(s)]\!] \stackrel{[\![s]\!]}{\to} [\![T_1]\!]$$

This is the same as

$$[\![FV(s[t/x])]\!] \stackrel{\langle \pi, [\![t]\!] \rangle}{\to} [\![FV(s) \setminus \{x\}]\!] \times [\![T_2]\!] \stackrel{[\![\lambda x.s]\!] \times \mathrm{id}}{\longrightarrow} [\![T_2 \Rightarrow T_1]\!] \times [\![T_2]\!] \stackrel{\varepsilon}{\to} [\![T_1]\!]$$

which is

$$[\![FV((\lambda x.s)t)]\!] \stackrel{[\![((\lambda x.s)t)]\!]}{\longrightarrow} [\![T_1]\!]$$

■

There is also a *completeness theorem*: if a judgement $\Gamma | t = s{:}T$ is true in all possible interpretations, then every theory (in a language this judgement is in) contains it.

The relevant construction is that of a syntactic cartesian closed category out of a theory, and an interpretation into it which makes exactly true the judgements in the theory. The curious reader can find the, somewhat laborious, treatment in Lambek & Scott's "Higher Order Categorical Logic".

7.3 Representation of primitive recursive functions in ccc's with natural numbers object

Dedekind observed, that in Set, the set ω is characterized by the following property: given any set X, any element $x \in X$ and any function $X \stackrel{f}{\to} X$, there is a unique function $F : \omega \to X$ such that $F(0) = x$ and $F(x+1) = f(F(x))$.

Lawvere took this up, and proposed this *categorical* property as a definition (in a more general context) of a "natural numbers object" in a category.

Definition 7.6 *In a category $\mathcal{C}$ with terminal object 1, a* natural numbers object *is a triple* $(0, N, S)$ *where N is an object of $\mathcal{C}$ and $1 \stackrel{0}{\to} N$, $N \stackrel{S}{\to} N$ arrows in $\mathcal{C}$, such that for any other such diagram*

$$1 \xrightarrow{x} X \xrightarrow{f} X$$

there is a unique *map $\phi : N \to X$ making*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "S"] \arrow[d, "\phi"] & N \arrow[d, "\phi"] \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

commute.

Of course we think of 0 as the zero element, and of S as the successor map. The defining property of a natural numbers object enables one to "do recursion", a weak version of which we show here: we show that every primitive recursive function can be represented in a ccc with natural numbers object.

Definition 7.7 *Let $\mathcal{C}$ be a ccc with natural numbers object $(0, N, S)$. We say that a number-theoretic function $F : \mathbb{N}^k \to \mathbb{N}$ is* represented *by an arrow $f : N^k \to N$ if for any k-tuple of natural numbers $n_1, \ldots n_k$, the diagram*

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "0"'] & N \arrow[r, "\langle S^{n_1},\ldots,S^{n_k}\rangle"] & N^k \arrow[d, "f"] \\
 & N \arrow[r, "S^{F(n_1,\ldots,n_k)}"'] & N
\end{tikzcd}$$

commutes.

Recall that the class of *primitive recursive* functions is given by the following clauses:

- The constant zero function $\lambda\vec{x}.0 : \mathbb{N}^k \to \mathbb{N}$, the function $\lambda x.x+1 : \mathbb{N} \to \mathbb{N}$ and the projections $\lambda\vec{x}.x_i : \mathbb{N} \to \mathbb{N}$ are primitive recursive;
- The primitive recursive functions are closed under composition: if $F_1, \ldots, F_k : \mathbb{N}^l \to \mathbb{N}$ and $G : \mathbb{N}^k \to \mathbb{N}$ are primitive recursive, then so is $G(\langle F_1, \ldots, F_k\rangle) : \mathbb{N}^l \to \mathbb{N}$;
- The primitive recursive functions are closed under *definition by primitive recursion*: if $G : \mathbb{N}^k \to \mathbb{N}$ and $H : \mathbb{N}^{k+2} \to \mathbb{N}$ are primitive recursive, and $F : \mathbb{N}^{k+1} \to \mathbb{N}$ is defined by $F(0, \vec{x}) = G(\vec{x})$ and $F(n+1, \vec{x}) = H(n, F(n, \vec{x}), \vec{x})$ then F is primitive recursive.

Proposition 7.8 *In a ccc $\mathcal{C}$ with natural numbers object, every primitive recursive function is representable.*

Proof. I do only the case for definition by primitive recursion. So by inductive hypothesis we have arrows G and H representing the homonymous functions. By interpretation of the λ-calculus, I use λ-terms: so there is an arrow

$$\lambda\vec{x}.G(\vec{x}) : 1 \to N^{(N^k)}$$

and an arrow

$$\lambda\vec{x}.H(n, \phi(\vec{x}), \vec{x}) : N^{(N^k)} \times N \to N^{(N^k)}$$

which is the interpretation of a term with free variables $\phi{:}N^{(N^k)}$ and $n{:}N$; this map is the exponential transpose of the map which intuitively sends $(n, \phi, \vec{x})$ to $(n, \phi(\vec{x}), \vec{x})$. Now look at

$$1 \xrightarrow{\langle\lambda\vec{x}.G(\vec{x}),0\rangle} N^{(N^k)} \times N \xrightarrow{(\lambda\vec{x}.H(n,\phi(\vec{x}),\vec{x}))\times S} N^{(N^k)} \times N$$

By the natural numbers object property, there is now a unique map

$$\bar{F} = \langle \tilde{F}, \sigma \rangle : N \to N^{(N^k)} \times N$$

which makes the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "\langle \lambda\vec{x}.G(\vec{x}),0\rangle"'] & N \arrow[r, "S"] \arrow[d, "\bar{F}"] & N \arrow[d, "\bar{F}"] \\
 & N^{(N^k)} \times N \arrow[r, "(\lambda\vec{x}.H(n,\phi(\vec{x}),\vec{x}))\times S"'] & N^{(N^k)} \times N
\end{tikzcd}$$

One verifies that σ is the identity, and that the composite

$$N^{k+1} \xrightarrow{\tilde{F}\times\mathrm{id}} N^{(N^k)} \times N^k \xrightarrow{\varepsilon} N$$

represents F. ∎

Exercise 129 Make these verifications.

One could ask: what is the class of those numerical functions (that is, functions $\mathbb{N}^k \to \mathbb{N}$) that are representable in every ccc with natural numbers object? It is not hard to see, that there are representable functions which are not primitive recursive (for example, the Ackermann function). On the other hand, Logic teaches us that every such representable function must be recursive, and that there are recursive, non-representable functions.

The answer is: the representable functions are precisely the so-called ε_0-*recursive functions* from Proof Theory; and this was essentially shown by Gödel in 1958.

8 Recursive Domain Equations

A *recursive domain equation* in a category $\mathcal{C}$ is an “equation” of the form:

$$X \cong F(X, \ldots, X)$$

where F is a functor: $(\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^m \to \mathcal{C}$.

Often, we are interested in not just any solution of such an equation, but in certain ‘universal’ solutions. Consider, as an example, the case $\mathcal{C} = \mathrm{Set}$, and $F(X) = 1 + (A \times X)$ for a fixed, nonempty set A. There are many solutions of $X \cong F(X)$ but one stands out: it is the set of *finite sequences of elements of* A.

In what sense is this a universal solution? Do such solutions always exist?

There is a piece of theory about this, which is by now a classic in theoretical Computer Science and was developed by Dana Scott around 1970; it is concerned with a certain subcategory of Pos. It is a nice application of the methods of category theory.

8.1 The category $\mathbb{CPO}$

Let $(P, \leq)$ be a partially ordered set. A *downset* or *downwards closed subset* of P is a subset $A \subseteq P$ such that if $a \in A$ and $p \leq a$, then $p \in A$. The *downwards closure* $\downarrow A$ of $A \subseteq P$ is the least downset of P containing A: $\downarrow A = \{p \in P \mid \exists a \in A. p \leq a\}$. We write $\downarrow p$ for $\downarrow \{p\}$.

An ω-*chain* in P is a function $f : \mathbb{N} \to P$ such that $f(0) \leq f(1) \leq \ldots$.
$(P, \leq)$ is a *cpo* or ω-*complete partial order* if every ω-chain in P has a colimit (i.e., least upper bound). This colimit is denoted $\bigsqcup_{n \in \mathbb{N}} f(n)$.

A monotone function $f : P \to Q$ between cpo’s is *continuous* if it preserves least upper bounds of ω-chains.

Exercise 130 Every cpo P can be regarded as a topological space, in the following way: open sets are those sets $A \subseteq P$ which are *upwards closed* ($a \in A \wedge a \leq b \Rightarrow b \in A$) and such that for any chain $f : \mathbb{N} \to P$, if $\bigsqcup_{n \in \mathbb{N}} f(n) \in A$ then $f(n) \in A$ for some $n \in \mathbb{N}$. Show that $f : P \to Q$ is continuous if and only if f is continuous w.r.t. the topology just defined.

There is a category $\mathbb{CPO}$ of cpo’s and continuous maps, and this category is our object of study for a while. Since every continuous function is monotone, there is a forgetful functor $U : \mathbb{CPO} \to \mathrm{Pos}$.

Theorem 8.1 *$U : \mathbb{CPO} \to \mathrm{Pos}$ is monadic.*

Proof. We have to show that U has a left adjoint $F : \mathrm{Pos} \to \mathbb{CPO}$ such that $\mathbb{CPO}$ is equivalent to the category of UF-algebras on Pos.

Call a subset A of a poset P an ω-*ideal* if there is an ω-chain $f : \mathbb{N} \to P$ such that A is the downwards closure of the image of f. Let $\omega{-}\mathrm{Idl}(P)$ the set of ω-ideals of P, ordered by inclusion. If $\varphi : P \to Q$ is a monotone map and

$A \subseteq P$ an ω-ideal, then $\downarrow\varphi[A] = \{q \in Q \mid \exists a \in A. q \leq \varphi(a)\}$ is also an ω-ideal of Q, for if $A = \downarrow\mathrm{im}(f)$ for $f : \mathbb{N} \to P$ then $\downarrow\varphi[A] = \downarrow\mathrm{im}(\varphi \circ f)$.

If $A_0 \subseteq A_1 \subseteq \ldots$ is an ω-chain of elements of $\omega\mathrm{-Idl}(P)$ then also $\bigcup_{n\in\mathbb{N}} A_n \in \omega\mathrm{-Idl}(P)$, for, if $A_i = \downarrow\mathrm{im}(f_i)$ define $f : \mathbb{N} \to P$ by:

$$f(n) = \begin{array}{l} f_n(m) \text{ where } m \text{ is minimal such that} \\ f_n(m) \text{ is an upper bound of} \\ \{f_i(k) \mid i, k \in \{0, \ldots, n\}\} \cup \{f(k) \mid k < n\} \end{array}$$

Then f is a chain and $\bigcup_{n\in\mathbb{N}} A_n = \downarrow\mathrm{im}(f)$.

So $\omega\mathrm{-Idl}(P)$ is a cpo; and since (for a monotone $\varphi : P \to Q$) the map $A \mapsto \downarrow\varphi[A]$ commutes with unions of ω-chains, it is a continuous map: $\omega\mathrm{-Idl}(P) \to \omega\mathrm{-Idl}(Q)$. So we have a functor $F : \mathrm{Pos} \to \mathbb{CPO}$: $F(P) = \omega\mathrm{-Idl}(P)$, and for $\varphi : P \to Q$ in Pos, $F(\varphi) : \omega\mathrm{-Idl}(P) \to \omega\mathrm{-Idl}(Q)$ is the map which sends A to $\downarrow\varphi[A]$.

Every monotone function $f : P \to U(Q)$ where Q is a cpo, gives a continuous function $\tilde{f} : \omega\mathrm{-Idl}(P) \to Q$ defined as follows: given $A \in \omega\mathrm{-Idl}(P)$, if $A = \downarrow\mathrm{im}(g)$ for a chain $g : \mathbb{N} \to P$, let $\tilde{f}(A)$ be the least upper bound in Q of the chain $f \circ g$. This is independent of the choice of g, for if $\downarrow\mathrm{im}(g) = \downarrow\mathrm{im}(g')$ then the chains $f \circ g$ and $f \circ g'$ have the same least upper bound in Q.

In the other direction, first let $\eta_P : P \to \omega\mathrm{-Idl}(P)$ be defined by $\eta_P(p) = \downarrow p$. Every continuous function $g : \omega\mathrm{-Idl}(P) \to Q$ gives a monotone $\bar{g} : P \to U(Q)$ by composition with η_P.

Exercise 131 Check that these two operations define a natural 1-1 correspondence between $\mathrm{Pos}(P, U(Q))$ and $\mathbb{CPO}(\omega\mathrm{-Idl}(P), Q)$ and therefore an adjunction $F \dashv U$ of which $\eta = (\eta_P)_P$ is the unit. What is the counit of this adjunction? Is it iso? Epi? What do you conclude about the functor U?

The monad UF has η as unit, and as multiplication

$$\mu = \bigcup : \omega\mathrm{-Idl}(\omega\mathrm{-Idl}(P)) \to \omega\mathrm{-Idl}(P)$$

taking the union of an ω-ideal of ω-ideals of P.

Exercise 132 Check that μ is well-defined. Prove that every UF-algebra is a cpo [Hint: compare with the proof that algebras for the powerset monad are equivalent to join-complete posets and join-preserving maps], so that $\mathbb{CPO}$ is equivalent to UF-Alg.

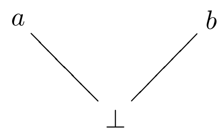
■

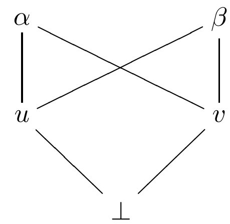
A corollary is now that $\mathbb{CPO}$ has all the limits that Pos has (that is, all small limits), and that these are created by the forgetful functor U. So, the limit in Pos of a diagram of cpo's and continuous maps, is also a cpo.

We shall also consider the category $\mathbb{CPO}_\perp$ of cpo's with a least element. $\mathbb{CPO}_\perp$ is a full subcategory of $\mathbb{CPO}$ (i.e., maps between objects of $\mathbb{CPO}_\perp$ are continuous but don't have to preserve the least element).

It is, categorically speaking, bad practice to require properties of objects without requiring the maps to preserve them. This is borne out by the fact that $\mathbb{CPO}_\perp$ loses the nice properties of $\mathbb{CPO}$:

Fact. $\mathbb{CPO}_\perp$ is neither finitely complete nor finitely cocomplete.
For instance consider the cpo's:

$T =$ 

and $U =$ 

Both T and U are objects of $\mathbb{CPO}_\perp$. Let $f, g : T \to U$ be defined by: $f(a) = g(a) = \alpha$, $f(b) = g(b) = \beta$, $f(\perp) = u$, $g(\perp) = v$. f and g are maps of $\mathbb{CPO}_\perp$, but cannot have an equalizer in $\mathbb{CPO}_\perp$.

Exercise 133 Prove this. Prove also that the coproduct of two one-element cpo's cannot exist in $\mathbb{CPO}_\perp$.

A map of cpo's with least elements which preserves the least element is called *strict*. The category $(\mathbb{CPO}_\perp)_s$ of cpo's with least element and strict continuous maps, is monadic over $\mathbb{CPO}$ by the "lifting monad": adding a least element (see chapter 6), and therefore complete.

Lemma 8.2 *Let P be a cpo and $(x_{ij})_{i,j\in\mathbb{N}}$ be a doubly indexed set of elements of P such that $i \le i'$ and $j \le j'$ implies $x_{ij} \le x_{i'j'}$. Then*

$$\bigsqcup_{i\in\mathbb{N}} \bigsqcup_{j\in\mathbb{N}} x_{ij} = \bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} x_{ij} = \bigsqcup_{i\in\mathbb{N}} x_{ii}$$

Exercise 134 Prove lemma 8.2.

Theorem 8.3 $\mathbb{CPO}$ *is cartesian closed.*

Proof. The exponent P^Q of two cpo's is the set of continuous maps from Q to P, ordered pointwise (i.e. $f \le g$ iff $\forall q \in Q.f(q) \le g(q)$). This is a cpo, because given a chain $f_0 \le f_1 \le \ldots$ of continuous maps, taking least upper bounds pointwise yields a continuous map:

$$f(q) = \bigsqcup_{i\in\mathbb{N}} f_i(q)$$

For, using lemma 8.2, $f(\bigsqcup_j q_j) = \bigsqcup_i \bigsqcup_j f_i(q_j) = \bigsqcup_j \bigsqcup_i f_i(q_j) = \bigsqcup_j f(q_j)$ ∎

Theorem 8.4 *Let P be a cpo with least element $\perp$. Then:*

a) *Every continuous map $f : P \to P$ has a least fixed point* $\mathrm{fix}(f)$ *(i.e. a least x with $f(x) = x$);*

b) *The assignment $f \mapsto \mathrm{fix}(f)$ is a continuous function: $P^P \to P$.*

Proof. Consider the chain $\bot \leq f(\bot) \leq f^2(\bot) \leq \ldots$. It's a chain because f is monotone. Let a be its least upper bound in P. Since f is continuous, $f(a) = f(\bigsqcup_{n\in\mathbb{N}} f^n(\bot)) = \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\bot) = a$, so a is a fixed point; if b is another fixed point of f then since $\bot \leq b = f(b)$, $f^n(\bot) \leq b$ for all n, hence $a \leq b$, so a is the least fixed point of f.

For the second statement, first notice that if $f \leq g$ in P^P then $f^n(\bot) \leq g^n(\bot)$ so $\mathrm{fix}(f) \leq \mathrm{fix}(g)$, so fix is monotone; and if $f_1 \leq f_2 \leq \ldots$ then $\mathrm{fix}(\bigsqcup_n f_n) = \bigsqcup_i (\bigsqcup_n f_n)^i(\bot) \geq \bigsqcup_i \bigsqcup_n f_n^i(\bot) =$ (by lemma 8.2) $\bigsqcup_n \bigsqcup_i f_n^i(\bot) = \bigsqcup_n \mathrm{fix}(f_n)$. The other inequality follows from the monotonicity of fix. ∎

For purposes of interpretation of recursion equations, it is convenient to have a notation for fixed points of functions of more than one variable. Let P and Q be cpo's with $\bot$ and $f : P \times Q \to P$ continuous; by cartesian closedness of $\mathbb{CPO}$ we have $\tilde{f} : Q \to P^P$ and we can consider the composite

$$Q \xrightarrow{\tilde{f}} P^P \xrightarrow{\mathrm{fix}} P$$

which is a continuous map by theorem 8.4. $\mathrm{fix}(\tilde{f}(q))$ is the least fixed point of the function which sends p to $f(p, q)$; we write $\mu p.f(p, q)$ for this.

Békic' theorem[1] says that if we want to find a *simultaneous* fixed point $(x, y) \in P \times Q$ of a function $f : P \times Q \to P \times Q$, we can do it in two steps, each involving a single fixed point calculation:

Theorem 8.5 (Békic's simultaneous fixed point theorem) *Let P and Q be cpo's with $\bot$ and $f : P \times Q \to P$, $g : P \times Q \to Q$ be continuous maps. Then the least fixed point of the map $\langle f, g\rangle : P \times Q \to P \times Q$ is the pair $(\hat{x}, \hat{y}) \in P \times Q$, where*

$$\begin{aligned} \hat{x} &= \mu x.f(x, \mu y.g(x, y)) \\ \hat{y} &= \mu y.g(\hat{x}, y) \end{aligned}$$

Proof. The least fixed point a of a function f has the property, that for any y, if $f(y) \leq y$ then $a \leq y$ (check this!), and moreover it is *characterized* by this property. Therefore the element $\mu x.\Phi(x, \vec{y})$ satisfies the *rule*:

$$\Phi(x', \vec{y}) \leq x' \Rightarrow \mu x.\Phi(x, \vec{y}) \leq x'$$

and is characterized by it. Now suppose:

$$\begin{array}{ll} (1) & f(a, b) \leq a \\ (2) & g(a, b) \leq b \end{array}$$

From (2) and the rule we get $\mu y.g(a, y) \leq b$, hence by (1) and monotonicity of f, $f(a, \mu y.g(a, y)) \leq f(a, b) \leq a$. Applying the rule again yields:

$$\hat{x} = \mu x.f(a, \mu y.g(x, y)) \leq a$$

[1] This theorem is known in recursion theory as *Smullyan's double recursion theorem*

so by (2) and monotonicity of g: $g(\hat{x},b) \leq g(a,b) \leq b$, so by the rule, $\mu y.g(\hat{x},y) \leq b$. We have derived that $(\hat{x},\hat{y}) \leq (a,b)$ from the assumption that $\langle f,g\rangle(a,b) \leq (a,b)$; this characterizes the least fixed point of $\langle f,g\rangle$, which is therefore $(\hat{x},\hat{y})$. ∎

Exercise 135 Generalize theorem 8.5 to 3 continuous functions with 3 variables, and, if you have the courage, to n continuous functions with n variables.

Exercise 136 Suppose D, E are cpo's with $\perp$ and $f : D \to E$, $g : E \to D$ continuous. Show that $\mu d.gf(d) = g(\mu e.fg(e))$ [Hint: use the rule given in the proof of theorem 8.5]

8.2 The category of cpo's with $\perp$ and embedding-projection pairs; limit-colimit coincidence; recursive domain equations

So far, we have seen that the category $\mathbb{CPO}$ is cartesian closed and complete. About $\mathbb{CPO}_\perp$ we can say that:

- $\mathbb{CPO}_\perp$ has products and the inclusion $\mathbb{CPO}_\perp \to \mathbb{CPO}$ preserves them;
- if Y has $\perp$ then Y^X has $\perp$, for any X.

So, also $\mathbb{CPO}_\perp$ is cartesian closed and supports therefore interpretation of simply typed λ-calculus (see chapter 7) and recursion (by the fixed point property). However, the structure of cpo's is much richer than that. First, we shall see that by restricting the morphisms of $\mathbb{CPO}_\perp$ we get a "cpo of cpo's". This will then, later, allow us to solve recursive domain equations like:

$$\begin{array}{cl} X \cong 1 + A \times X & \text{lists on alphabet } A \\ X \cong X^X & \text{untyped } \lambda\text{-calculus} \end{array}$$

First we have to go through some technique.

Definition 8.6 *Let P and Q posets. A pair* $P \underset{i}{\overset{r}{\leftrightarrows}} Q$ *of monotone maps is called an* embedding-projection pair *(e-p pair for short), where i is the embedding, r the projection, if $i \dashv r$ and i is full and faithful; equivalently: $ri = \mathrm{id}_P$ and $ir \leq \mathrm{id}_Q$.*

By uniqueness of adjoints, each member of an e-p pair determines the other. It is evident that $\langle \mathrm{id}_P, \mathrm{id}_P\rangle$ is an e-p pair, and that e-p pairs compose. We can therefore define a category $\mathbb{CPO}_\perp^{\mathrm{EP}}$: objects are cpo's with $\perp$, and morphisms $P \to Q$ are e-p pairs $P \underset{i}{\overset{r}{\leftrightarrows}} Q$ such that both i and r are continuous.

Lemma 8.7 *Let* $P \underset{i}{\overset{r}{\leftrightarrows}} Q$ *be an e-p pair, where P and Q are cpo's with $\perp$. Then both i and r are strict, and i is continuous.*

Proof. Being a left adjoint, i preserves colimits, so i is strict and continuous; since $ri = \mathrm{id}_P$ we also have $r(\bot_Q) = ri(\bot_P) = \bot_P$. ∎

For the following theorem, recall that every diagram in $\mathbb{CPO}_\bot$ with *strict* continuous maps will have a limit in $\mathbb{CPO}_\bot$, since this takes place in $(\mathbb{CPO}_\bot)_s$ which is monadic over $\mathbb{CPO}$.

Theorem 8.8 ($\mathbb{CPO}_\bot^{\mathrm{EP}}$ as "cpo of cpo's"; limit-colimit coincidence)

a) *Any chain of maps*

$$P_1 \underset{i_1}{\overset{r_1}{\leftrightarrows}} P_2 \underset{i_2}{\overset{r_2}{\leftrightarrows}} P_3 \underset{i_3}{\overset{r_3}{\leftrightarrows}} \cdots$$

in $\mathbb{CPO}_\bot^{\mathrm{EP}}$ has a colimit in $\mathbb{CPO}_\bot^{\mathrm{EP}}$;

b) *the vertex of this colimit, P, is the limit (in $\mathbb{CPO}_\bot$) of the diagram*

$$P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} cdots$$

and P is also the colimit in $\mathbb{CPO}_\bot$ of the diagram

$$P_1 \xrightarrow{i_1} P_2 \xrightarrow{i_2} \cdots$$

Proof. We prove a) and b) simultaneously. Note that the limit of $P_1 \overset{r_1}{\leftarrow} P_2 \overset{r_2}{\leftarrow} \ldots$ exists in $\mathbb{CPO}_\bot$ since all maps are strict by lemma 8.7; it is the object $P = \{(p_1, p_2, \ldots) \in \prod_{n\geq 1} P_n \,|\, \forall i \geq 1\, r_i(p_{i+1}) = p_i\}$ with pointwise order.

For any k we have maps $P_k \underset{e_k}{\overset{\pi_k}{\leftrightarrows}} P$ where π_k is the k-th projection, and e_k is defined by:

$$(e_k(p))_j = \begin{cases} r_j r_{j+1} \cdots r_{k_1}(p) & \text{if } j < k \\ p & \text{if } j = k \\ i_{j_1} \cdots i_k(p) & \text{if } j > k \end{cases}$$

Now $\pi_k e_k(p) = (e_k(p))_k = p$ and

$$\begin{array}{rcl} e_k\pi_k(p_1, p_2, \ldots) & = & (p_1, p_2, \ldots, p_k, i_k r_k(p_{k+1}), i_{k+1} i_k r_k r_{k+1}(p_{k+2}), \ldots) \\ & \leq & (p_1, p_2, \ldots) \end{array}$$

So $\langle e_k, \pi_k\rangle$ is an e-p pair. Since obviously $r_i\pi_{i+1} = \pi_i$ for all $i \in \mathbb{N}$, also $e_{k+1} i_k = e_k$ must hold (since one component of an e-p pair uniquely determines the other), hence

$$\{P_k \overset{\langle e_k, \pi_k\rangle}{\to} P \,|\, k \geq 1\}$$

is a cocone in $\mathbb{CPO}_\bot^{\mathrm{EP}}$ for the given chain.

Suppose now that $\{P_k \overset{\langle j_k, s_k\rangle}{\to} Q \,|\, k \geq 1\}$ is another cocone for the chain in $\mathbb{CPO}_\bot^{\mathrm{EP}}$. Immediately, we have (since P is the limit of $P_1 \overset{r_1}{\leftarrow} P_2 \overset{r_2}{\leftarrow} \cdots$) a unique

$\sigma : Q \to P$ such that $s_k = \pi_k \sigma$ for all k; $\sigma(q) = (s_1(q), s_2(q), \ldots)$. Note that σ is continuous. Since we have a cocone, for any $(p_1, p_2, \ldots)$ in P we have that in Q:

$$j_k(p_k) = j_{k+1} i_k(p_k) = j_{k+1} i_k r_k(p_{k+1}) \le j_{k+1}(p_{k+1})$$

so $j_1(p_1) \le j_2(p_2) \le \ldots$ and we define $J : P \to Q$ by

$$J(p_1, p_2, \ldots) = \bigsqcup_k j_k(p_k)$$

Then $J(\sigma(q)) = \bigsqcup_k j_k s_k(q) \le q$ because $\langle j_k, s_k \rangle$ is an e-p pair, and by continuity of s_n,

$$\begin{array}{rcl} (\sigma J(p_1, p_2, \ldots))_n & = & s_n(\bigsqcup_k j_k(p_k)) \quad = \\ s_n(\bigsqcup_{k \ge n} j_k(p_k)) & = & \bigsqcup_{k \ge n} s_n j_k(p_k) \end{array}$$

For $k \ge n$ write $\langle i_{nk}, r_{nk} \rangle$ for the e-p pair $P_n \rightleftarrows P_k$. Using that $r_{nk} s_k = s_n$, $s_k j_k = \mathrm{id}_{P_k}$,

$$\begin{array}{rcl} \bigsqcup_{k \ge n} s_n j_k(p_k) & = & \bigsqcup_{k \ge n} r_{nk} s_k j_k(p_k) \quad = \\ \bigsqcup_{k \ge n} r_{nk}(p_k) & = & \bigsqcup_{k \ge n} p_n \quad = p_n \end{array}$$

So $\sigma J = \mathrm{id}_P$; i.e. the cocone with vertex Q factors uniquely through the one with vertex P; hence the latter is colimiting.

The only thing which remains to be proven, is that $\{P_k \xrightarrow{e_k} P \mid k \ge 1\}$ is also a colimiting cocone in $\mathbb{CPO}_\perp$. Firstly, from the definition of $P_k \underset{e_k}{\overset{\pi_k}{\rightleftarrows}} P$ and the already seen

$$e_k \pi_k(p_1, p_2, \ldots) = (p_1, \ldots, p_k, i_k r_k(p_{k+1}), i_{k+1} i_k r_k r_{k+1}(p_{k+2}), \ldots)$$

it is immediate that, in P,

$$(p_1, p_2, \ldots) = \bigsqcup_{k \ge 1} e_k \pi_k(p_1, p_2, \ldots) = \bigsqcup_{k \ge 1} e_k(p_k)$$

So if $\{P_k \xrightarrow{f_k} Q \mid k \ge 1\}$ is another cocone in $\mathbb{CPO}_\perp$ we can define a continuous factorization $P \xrightarrow{f} Q$ by

$$f(p_1, p_2, \ldots) = \bigsqcup_k f_k(p_k)$$

but in fact we have no other choice, hence the factorization is unique. ∎

Define an ω-*category* as a category where every chain of maps

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \to \cdots$$

has a colimiting cocone; and call a functor between ω-categories *continuous* if it preserves colimits of chains. Theorem 8.8 says that $\mathbb{CPO}_\perp^{\mathrm{EP}}$ is an ω-category.

Lemma 8.9 *Let $\mathcal{A}$ be an ω-category and $F : \mathcal{A} \to \mathcal{A}$ continuous. If $A \in \mathcal{A}_0$ and $A \xrightarrow{f} F(A)$ a map, and the chain*

$$A \xrightarrow{f} F(A) \xrightarrow{F(f)} F^2(A) \xrightarrow{F^2(f)} F^3(A) \to \dots$$

has colimit with vertex D, then D is isomorphic to $F(D)$.

In particular, if $\mathcal{A}$ has an initial object, F has an up to isomorphism fixed point.

Exercise 137 Prove lemma 8.9. It generalizes the idea of the fixed point property for cpo's.

For any endofunctor $F : \mathcal{A} \to \mathcal{A}$ we define the category F-Alg of F-algebras in a similar way as for a monad, but simpler (since the functor has less structure than the monad): objects are maps $FX \xrightarrow{h} X$, just like that, and maps $(X, h) \to (Y, k)$ are maps $f : X \to Y$ in $\mathcal{A}$ such that

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "h"'] & F(Y) \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. We have:

Lemma 8.10 (Lambek's Lemma) *If $F(X) \xrightarrow{h} X$ is an initial object of F-Alg, then h is an isomorphism in $\mathcal{A}$.*

Proof. $F^2(X) \xrightarrow{F(h)} F(X)$ is also an F-algebra, so there is a unique $k : X \to F(X)$ such that

$$\begin{tikzcd} F(X) \arrow[r, "F(k)"] \arrow[d, "h"'] & F^2(X) \arrow[d, "F(h)"] \\ X \arrow[r, "k"'] & F(X) \end{tikzcd}$$

commutes. Since also

$$\begin{tikzcd} F^2(X) \arrow[r, "F(h)"] \arrow[d, "F(h)"'] & F(X) \arrow[d, "h"] \\ F(X) \arrow[r, "h"'] & X \end{tikzcd}$$

commutes, h is a map of F-algebras: $(F(X), F(h)) \to (X, h)$. Therefore is $hk : (X, h) \to (X, h)$ a map in F-Alg and since (X, h) is initial, $hk = \mathrm{id}_X$. Then $kh = F(h)F(k) = F(hk) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, so h is iso with inverse k. ∎

Exercise 138 In the situation of lemma 8.9, i.e. $\mathcal{A}$ an ω-category, $F : \mathcal{A} \to \mathcal{A}$ continuous, the colimit of

$$0 \xrightarrow{!} F(0) \xrightarrow{F(!)} F^2(0) \xrightarrow{F^2(!)} F^3(0) \to \ldots$$

(where 0 is initial in $\mathcal{A}$, and ! the unique map $0 \to F(0)$) gives the initial F-algebra.

In view of Lambek's Lemma and other considerations (such as the desirability of induction priciples for elements of recursively defined domains), we aim to solve an equation:

$$X \cong F(X)$$

as an initial F-algebra. So we have seen, that as long as F is a continuous functor, we do have initial F-algebras in $\mathbb{CPO}_\perp^{\mathrm{EP}}$. But this in itself did not require the introduction of $\mathbb{CPO}_\perp^{\mathrm{EP}}$ for also $\mathbb{CPO}$ is an ω-category with an initial object (as is, by the way, Set). The force of the embedding-projection pairs resides in the possibilities of handling "mixed variance". Since the expression X^X does not define a functor but in general, the expression X^Y defines a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, one says that in X^X, the variable X occurs both covariantly and contravariantly (or, positively and negatively). We shall see that functors of mixed variance on $\mathbb{CPO}_\perp$, that is: functors $(\mathbb{CPO}_\perp^{\mathrm{op}})^n \times (\mathbb{CPO}_\perp)^m \to \mathbb{CPO}_\perp$, can, under certain conditions, be transformed into continuous covariant functors: $(\mathbb{CPO}_\perp^{\mathrm{EP}})^{n+m} \to \mathbb{CPO}_\perp^{\mathrm{EP}}$. Composition with the diagonal functor $\Delta : \mathbb{CPO}_\perp^{\mathrm{EP}} \to (\mathbb{CPO}_\perp^{\mathrm{EP}})^{n+m}$ gives a continuous endofunctor on $\mathbb{CPO}_\perp^{\mathrm{EP}}$ which has a fixed point (up to isomorphism).

The first ingredient we need is the notion of *local continuity*. Recall that in the proof that $\mathbb{CPO}$ was a ccc, we have basically seen that for cpo's P and Q the set $\mathbb{CPO}(P, Q)$ is itself a cpo. Of course, this holds for $\mathbb{CPO}^{\mathrm{op}}$ too, and also for products of copies of $\mathbb{CPO}$ and $\mathbb{CPO}^{\mathrm{op}}$:

$$(\mathbb{CPO}^{\mathrm{op}})^n \times (\mathbb{CPO})^m((A'_1, \ldots, A'_n, B_1, \ldots, B_m), (A_1, \ldots, A_n, B'_1, \ldots, B'_m))$$

is the cpo $\mathbb{CPO}(A_1, A'_1) \times \cdots \times \mathbb{CPO}(B_m, B'_m)$.

Definition 8.11 *A functor $F : (\mathbb{CPO}^{\mathrm{op}})^n \times (\mathbb{CPO})^m \to \mathbb{CPO}$ is called* locally continuous *if its action on maps:*

$$F_1 : (\mathbb{CPO}^{\mathrm{op}})^n \times (\mathbb{CPO})^m((\vec{A}', \vec{B}), (\vec{A}, \vec{B}')) \to \mathbb{CPO}(F(\vec{A}', \vec{B}), F(\vec{A}, \vec{B}'))$$

is a map of cpo's, that is: continuous. We have the same notion if we replace $\mathbb{CPO}$ by $\mathbb{CPO}_\perp$.

Example. The product and coproduct functors: $\mathbb{CPO} \times \mathbb{CPO} \to \mathbb{CPO}$, and the exponent functor: $\mathbb{CPO}^{\mathrm{op}} \times \mathbb{CPO} \to \mathbb{CPO}$ are locally continuous.

Theorem 8.12 *Suppose $F : \mathbb{CPO}_\perp^{\mathrm{op}} \times \mathbb{CPO}_\perp \to \mathbb{CPO}_\perp$ is locally continuous. Then there is an $\hat{F} : \mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}} \to \mathbb{CPO}_\perp^{\mathrm{EP}}$ which is continuous and has the same action on objects as F.*

Proof. We put $\hat{F}(P,Q) = F(P,Q)$, so the last statement of the theorem has been taken care of. Given e-p pairs $P \underset{i}{\overset{r}{\leftrightarrows}} P'$ and $Q \underset{j}{\overset{s}{\leftrightarrows}} Q'$ we have an e-p pair $F(P,Q) \underset{F(r,j)}{\overset{F(i,s)}{\leftrightarrows}} F(P',Q')$ since $F(i,s) \circ F(r,j) = F(r \circ i, s \circ j)$ (recall that F is contravariant in its first argument!) $= F(\mathrm{id},\mathrm{id}) = \mathrm{id}_{F(P,Q)}$, and $F(r,j) \circ F(i,s) = F(i \circ r, j \circ s) \leq F(\mathrm{id},\mathrm{id}) = \mathrm{id}_{F(P',Q')}$ (the last inequality is by local continuity of F). So we let

$$\hat{F}(\langle i,r\rangle, \langle j,s\rangle) = \langle F(r,j), F(i,s)\rangle$$

Then clearly, $\hat{F}$ is a functor. To see that $\hat{F}$ is continuous, suppose we have a chain of maps in $\mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}}$: $(A_1,B_1) \to (A_2,B_2) \to \ldots$ with colimit (D,E). That means in $\mathbb{CPO}_\perp^{\mathrm{EP}}$ we have two chains, each with its colimit:

$$\begin{tikzcd} A_1 \arrow[r] \arrow[dr, "\langle i^1,r^1\rangle"'] & A_2 \arrow[r] \arrow[d, "\langle i^2,r^2\rangle\cdots"] & \cdots \\ & D & \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[r] \arrow[dr, "\langle j^1,s^1\rangle"'] & B_2 \arrow[r] \arrow[d, "\langle j^2,s^2\rangle\cdots"] & \cdots \\ & E & \end{tikzcd}$$

From the proof of theorem 8.8 we know that $\mathrm{id}_D = \bigsqcup_n i^n \circ r^n$ and $\mathrm{id}_E = \bigsqcup_n j^n \circ s^n$ so $\mathrm{id}_(D,E) = \bigsqcup(i^n \circ r^n, j^n \circ s^n)$. From local continuity of F then, $\mathrm{id}_{\hat{F}(D,E)} = \hat{F}(\mathrm{id}_{(D,E)}) = \langle F(\mathrm{id}_D,\mathrm{id}_E), F(\mathrm{id}_D,\mathrm{id}_E\rangle = \bigsqcup_n \langle F(i^n \circ r^n, j^n \circ s^n), F(i^n \circ r^n, j^n \circ s^n)\rangle$. But this characterizes the colimit of a chain in $\mathbb{CPO}_\perp^{\mathrm{EP}}$, so $\hat{F}(D,E) = F(D,E)$ is isomorphic to the vertex of the colimit in $\mathbb{CPO}_\perp^{\mathrm{EP}}$ of the chain:

$$\hat{F}(A_1,B_1) \to \hat{F}(A_2,B_2) \to \cdots$$

∎

Example: a model of the untyped λ-calculus. In the *untyped* λ-calculus, we have a similar formalism as the one given in Chapter 7, but now there are no types. That means, that variables denote functions and arguments at the same time!

In order to model this, we seek a nontrivial solution to:

$$X \cong X^X$$

(Since $X = 1$ is always a solution, “nontrivial” means: not this one) According to theorem 8.12, the exponential functor $\mathbb{CPO}_\perp^{\mathrm{op}} \times \mathbb{CPO}_\perp \to \mathbb{CPO}_\perp$, which sends (X,Y) to Y^X and is locally continuous, gives rise to a continuous functor $\mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}} \overset{\mathrm{Exp}}{\to} \mathbb{CPO}_\perp^{\mathrm{EP}}$. Let $F : \mathbb{CPO}_\perp^{\mathrm{EP}} \to \mathbb{CPO}_\perp^{\mathrm{EP}}$ be the composite

$$\mathbb{CPO}_\perp^{\mathrm{EP}} \overset{\Delta}{\to} \mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}} \overset{\mathrm{Exp}}{\to} \mathbb{CPO}_\perp^{\mathrm{EP}}$$

Δ (the diagonal functor) is continuous, so F is, since continuous functors compose.

Exercise 139 Show that the functor F works as follows: for an e-p pair $P \underset{i}{\overset{r}{\leftrightarrows}} Q$,

$F(\langle i, r\rangle)$ is the e-p pair $P^P \underset{I}{\overset{R}{\leftrightarrows}} Q^Q$ where $I(f) = i \circ f \circ r$ and $R(g) = r \circ g \circ i$.

Let us try and apply lemma 8.9; we need a non-initial object P and a map $P \xrightarrow{f} F(P)$ in $\mathbb{CPO}_{\perp}^{\mathrm{EP}}$.

That is, an embedding-projection pair $P \leftrightarrows P^P$. Well, it is readily checked that for *any* cpo P with $\perp$ the pair $\langle \iota, \rho\rangle$ is an e-p pair, where $\iota : P \to P^P$ sends p to the function which has constant value p, and $\rho : P^P \to P$ sends the function f to $f(\perp)$. We now summarize all our preparations into the final theorem.

Theorem 8.13 (Scott) *Let P be a cpo with $\perp$. Define a diagram*

$$D_0 \underset{i_0}{\overset{r_0}{\leftrightarrows}} D_1 \underset{i_1}{\overset{r_1}{\leftrightarrows}} D_2 \underset{i_2}{\overset{r_2}{\leftrightarrows}} \cdots$$

in $\mathbb{CPO}$ as follows:

- $D_0 = P$; $D_{n+1} = D_n^{D_n}$;
- $i_0 : P \to P^P$ *is* $\lambda p.\lambda q.p$;
- $r_0 : P^P \to P$ *is* $\lambda f.f(\perp)$;
- $i_{n+1} : D_n^{D_n} = D_{n+1} \to D_{n+1}^{D_{n+1}}$ *is* $\lambda f.i_n \circ f \circ r_n$;
- $r_{n+1} : D_{n+1}^{D_{n+1}} \to D_n^{D_n}$ *is* $\lambda g.r_n \circ g \circ i_n$.

Let D_∞ be (vertex of) the limit in $\mathbb{CPO}$ of

$$D_0 \overset{r_0}{\leftarrow} D_1 \overset{r_1}{\leftarrow} D_2 \overset{r_2}{\leftarrow} \cdots$$

Then $D_\infty \cong D_\infty^{D_\infty}$ in $\mathbb{CPO}$.

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