Basic Category Theory and Topos Theory

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1 Categories and Functors

1.1 Definitions and examples

A *category* $\mathcal{C}$ is given by a collection $\mathcal{C}_0$ of *objects* and a collection $\mathcal{C}_1$ of *arrows* which have the following structure.

- Each arrow has a *domain* and a *codomain* which are objects; one writes $f : X \to Y$ or $X \xrightarrow{f} Y$ if X is the domain of the arrow f, and Y its codomain. One also writes $X = \mathrm{dom}(f)$ and $Y = \mathrm{cod}(f)$;
- Given two arrows f and g such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, the *composition* of f and g, written gf, is defined and has domain $\mathrm{dom}(f)$ and codomain $\mathrm{cod}(g)$:
$$(X \xrightarrow{f} Y \xrightarrow{g} Z) \mapsto (X \xrightarrow{gf} Z)$$
- Composition is *associative*, that is: given $f : X \to Y$, $g : Y \to Z$ and $h : Z \to W$, $h(gf) = (hg)f$;
- For every object X there is an *identity* arrow $\mathrm{id}_X : X \to X$, satisfying $\mathrm{id}_X g = g$ for every $g : Y \to X$ and $f\mathrm{id}_X = f$ for every $f : X \to Y$.

Exercise 1 Show that id_X is the *unique* arrow with domain X and codomain X with this property.

Instead of "arrow" we also use the terms "morphism" or "map".

Examples

a) **1** is the category with one object $*$ and one arrow, id_*;

b) **0** is the empty category. It has no objects and no arrows.

c) A *preorder* is a set X together with a binary relation $\leq$ which is reflexive (i.e. $x \leq x$ for all $x \in X$) and transitive (i.e. $x \leq y$ and $y \leq z$ imply $x \leq z$ for all $x, y, z \in X$). This can be viewed as a category, with set of objects X and for every pair of objects (x, y) such that $x \leq y$, exactly one arrow: $x \to y$.

Exercise 2 Prove this. Prove also the converse: if $\mathcal{C}$ is a category such that $\mathcal{C}_0$ is a set, and such that for any two objects X, Y of $\mathcal{C}$ there is at most one arrow: $X \to Y$, then $\mathcal{C}_0$ is a preordered set.

d) A *monoid* is a set X together with a binary operation, written like multiplication: xy for $x, y \in X$, which is associative and has a unit element $e \in X$, satisfying $ex = xe = x$ for all $x \in X$. Such a monoid is a category with one object, and an arrow x for every $x \in X$.

e) Set is the category which has the class of all sets as objects, and functions between sets as arrows.

Most basic categories have as objects certain mathematical structures, and the structure-preserving functions as morphisms. Examples:

f) Top is the category of topological spaces and continuous functions.

g) Grp is the category of groups and group homomorphisms.

h) Rng is the category of rings and ring homomorphisms.

i) Grph is the category of graphs and graph homomorphisms.

j) Pos is the category of partially ordered sets and monotone functions.

Given two categories C and D, a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of operations $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$ and $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$, such that for each $f : X \to Y$, $F_1(f) : F_0(X) \to F_0(Y)$ and:

- for $X \stackrel{f}{\to} Y \stackrel{g}{\to} Z$, $F_1(gf) = F_1(g)F_1(f)$;
- $F_1(\mathrm{id}_X) = \mathrm{id}_{F_0(X)}$ for each $X \in \mathcal{C}_0$.

But usually we write just F instead of F_0, F_1.

Examples.

a) There is a functor U : Top $\to$ Set which assigns to any topological space X its underlying set. We call this functor "forgetful": it "forgets" the mathematical structure. Similarly, there are forgetful functors Grp $\to$ Set, Grph $\to$ Set, Rng $\to$ Set, Pos $\to$ Set etcetera;

b) For every category C there is a unique functor $\mathcal{C} \to \mathbf{1}$ and a unique one $\mathbf{0} \to \mathcal{C}$;

c) Given two categories C and D we can define the *product category* $\mathcal{C} \times \mathcal{D}$ which has as objects pairs $(C, D) \in \mathcal{C}_0 \times \mathcal{D}_0$, and as arrows:$(C, D) \to (C', D')$ pairs (f, g) with $f : C \to C'$ in C, and $g : D \to D'$ in D. There are functors $\pi_0 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$;

d) Given two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ one can define the composition $GF : \mathcal{C} \to \mathcal{E}$. This composition is of course associative and since we have, for any category $\mathcal{C}$, the *identity functor* $\mathcal{C} \to \mathcal{C}$, we have a category Cat which has categories as objects and functors as morphisms.

e) Given a set A, consider the set $\tilde{A}$ of strings $a_1 \ldots a_n$ on the alphabet $A \cup A^{-1}$ (A^{-1} consists of elements a^{-1} for each element a of A; the sets A and A^{-1} are disjoint and in 1-1 correspondence with each other), such that for no $x \in A$, xx^{-1} or $x^{-1}x$ is a substring of $a_1 \ldots a_n$. Given two such strings $\vec{a} = a_1 \ldots a_n, \vec{b} = b_1 \ldots b_m$, let $\vec{a} \star \vec{b}$ the string formed by first taking $a_1 \ldots a_n b_1 \ldots b_m$ and then removing from this string, successively, substrings of form xx^{-1} or $x^{-1}x$, until one has an element of $\tilde{A}$.

   This defines a group structure on $\tilde{A}$. $\tilde{A}$ is called the *free group* on the set A.

Exercise 3 Prove this, and prove that the assignment $A \mapsto \tilde{A}$ is part of a functor: Set $\to$ Grp. This functor is called the *free functor*.

f) Every directed graph can be made into a category as follows: the objects are the vertices of the graph and the arrows are paths in the graph. This defines a functor from the category Dgrph of directed graphs to Cat. The image of a directed graph D under this functor is called the category *generated* by the graph D.

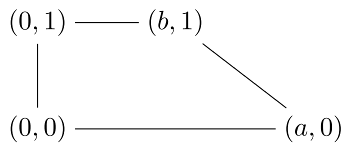
g) **Quotient categories**. Given a category C, a *congruence relation* on $\mathcal{C}$ specifies, for each pair of objects X, Y, an equivalence relation $\sim_{X,Y}$ on the class of arrows $\mathcal{C}(X,Y)$ which have domain X and codomain Y, such that

- for $f, g : X \to Y$ and $h : Y \to Z$, if $f \sim_{X,Y} g$ then $hf \sim_{X,Z} hg$;
- for $f : X \to Y$ and $g, h : Y \to Z$, if $g \sim_{Y,Z} h$ then $gf \sim_{X,Z} hf$.

Given such a congruence relation $\sim$ on $\mathcal{C}$, one can form the quotient category $\mathcal{C}/\sim$ which has the same objects as $\mathcal{C}$, and arrows $X \to Y$ are $\sim_{X,Y}$-equivalence classes of arrows $X \to Y$ in C.

Exercise 4 Show this and show that there is a functor $\mathcal{C} \to \mathcal{C}/\sim$, which takes each arrow of $\mathcal{C}$ to its equivalence class.

h) An example of this is the following ("homotopy"). Given a topological space X and points $x, y \in X$, a *path* from x to y is a continuous mapping f from some closed interval $[0, a]$ to X with $f(0) = x$ and $f(a) = y$. If $f : [0, a] \to X$ is a path from x to y and $g : [0, b] \to X$ is a path from y to z there is a path $gf : [0, a+b] \to X$ (defined by $gf(t) = \left\{ \begin{array}{ll} f(t) & t \leq a \\ g(t-a) & \text{else} \end{array} \right)$ from x to z. This makes X into a category, the *path category* of X, and of course this also defines a functor Top $\to$ Cat. Now given paths $f : [0, a] \to X$, $g : [0, b] \to X$, both from x to y, one can define $f \sim_{x,y} g$ if there is a continuous map $F : A \to X$ where A is the area:



in $\mathbb{R}^2$, such that

$$\begin{array}{lll} F(t,0) = & f(t) & \\ F(t,1) = & g(t) & \\ F(0,s) = & x & s \in [0,1] \\ F(s,t) = & y & (s,t) \text{ on the segment } (b,1) - (a,0) \end{array}$$

One can easily show that this is a congruence relation. The quotient of the path category by this congruence relation is a category called the category of *homotopy classes* of paths in X.

i) let $\mathcal{C}$ be a category such that for every pair (X, Y) of objects the collection $\mathcal{C}(X, Y)$ of arrows from X to Y is a set (such $\mathcal{C}$ is called *locally small*).

For any object C of $\mathcal{C}$ then, there is a functor $y_C : \mathcal{C} \to \mathrm{Set}$ which assigns to any object C' the set $\mathcal{C}(C, C')$. Any arrow $f : C' \to C''$ gives by composition a function $\mathcal{C}(C, C') \to \mathcal{C}(C, C'')$, so we have a functor. A functor of this form is called a *representable functor*.

j) Let $\mathcal{C}$ be a category and C an object of $\mathcal{C}$. The *slice category* $\mathcal{C}/C$ has as objects all arrows g which have codomain C. An arrow from $g : D \to C$ to $h : E \to C$ in $\mathcal{C}/C$ is an arrow $k : D \to E$ in $\mathcal{C}$ such that $hk = g$. Draw like:

$$\begin{tikzcd} D \arrow[rr, "k"] \arrow[dr, "g"'] & & E \arrow[dl, "h"] \\ & C & \end{tikzcd}$$

We say that *this diagram commutes* if we mean that $hk = g$.

Exercise 5 Convince yourself that the assignment $C \mapsto \mathcal{C}/C$ gives rise to a functor $\mathcal{C} \to \mathrm{Cat}$.

k) Recall that for every group $(G, \cdot)$ we can form a group $(G, \star)$ by putting $f \star g = g \cdot f$.

For categories the same construction is available: given $\mathcal{C}$ we can form a category $\mathcal{C}^{\mathrm{op}}$ which has the same objects and arrows as C, but with reversed direction; so if $f : X \to Y$ in C then $f : Y \to X$ in $\mathcal{C}^{\mathrm{op}}$. To make it notationally clear, write $\bar{f}$ for the arrow $Y \to X$ corresponding to $f : X \to Y$ in C. Composition in $\mathcal{C}^{\mathrm{op}}$ is defined by:

$$\bar{f}\bar{g} = \overline{gf}$$

Often one reads the term "contravariant functor" in the literature. What I call functor, is then called "covariant functor". A contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ inverts the direction of the arrows, so $F_1(f) : F_0(\mathrm{cod}(f)) \to F_0(\mathrm{dom}(f))$ for arrows f in $\mathcal{C}$. In other words, a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ (equivalently, from $\mathcal{C}$ to $\mathcal{D}^{\mathrm{op}}$).

Of course, any functor $F : \mathcal{C} \to \mathcal{D}$ gives a functor $F^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$. In fact, we have a functor $(-)^{\mathrm{op}} : \mathrm{Cat} \to \mathrm{Cat}$.

Exercise 6 Let $\mathcal{C}$ be locally small. Show that there is a functor (the "Hom functor") $\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$, assigning to the pair (A, B) of objects of $\mathcal{C}$, the set $\mathcal{C}(A, B)$.

l) Given a partially ordered set $(X, \leq)$ we make a topological space by defining $U \subseteq X$ to be open iff for all $x, y \in X$, $x \leq y$ and $x \in U$ imply $y \in U$ (U is "upwards closed", or an "upper set"). This is a topology, called the *Alexandroff topology* w.r.t. the order $\leq$.

If $(X, \leq)$ and $(Y, \leq)$ are two partially ordered sets, a function $f : X \to Y$ is monotone for the orderings if and only if f is continuous for the Alexandroff topologies. This gives an important functor: Pos $\to$ Top.

Exercise 7 Show that the construction of the quotient category in example g) generalizes that of a quotient group by a normal subgroup. That is, regard a group G as a category with one object; show that there is a bijection between congruence relations on G and normal subgroups of G, and that for a normal subgroup N of G, the quotient category by the congruence relation corresponding to N, is to the quotient group G/N.

m) "Abelianization". Let Abgp be the category of abelian groups and homomorphisms. For every group G the subgroup $[G, G]$ generated by all elements of form $aba^{-1}b^{-1}$ is a normal subgroup. $G/[G, G]$ is abelian, and for every group homomorphism $\phi : G \to H$ with H abelian, there is a unique homomorphism $\bar{\phi} : G/[G, G] \to H$ such that the diagram

$$\begin{tikzcd} & G \arrow[dl, "p"'] \arrow[dr, "\phi"] & \\ G/[G,G] \arrow[rr, "\bar{\phi}"'] & & H \end{tikzcd}$$

commutes. Show that this gives a functor: Grp $\to$ Abgp.

n) "Specialization ordering". Given a topological space X, you can define a preorder $\leq_s$ on X as follows: say $x \leq_s y$ if for all open sets U, if $x \in U$ then $y \in U$. $\leq_s$ is a partial order iff X is a T_0-space.

For many spaces, $\leq_s$ is trivial (in particular when X is T_1) but in case X is for example the Alexandroff topology on a poset $(X, \leq)$ as in l), then $x \leq_s y$ iff $x \leq y$.

Exercise 8 If $f : X \to Y$ is a continuous map of topological spaces then f is monotone w.r.t. the specialization orderings $\leq_s$. This defines a functor Top $\to$ Preord, where Preord is the category of preorders and monotone functions.

Exercise 9 Let X be the category defined as follows: objects are pairs (I, x) where I is an open interval in $\mathbb{R}$ and $x \in I$. Morphisms $(I, x) \to (J, y)$ are differentiable functions $f : I \to J$ such that $f(x) = y$.

Let Y be the (multiplicative) monoid $\mathbb{R}$, considered as a category. Show that the operation which sends an arrow $f : (I, x) \to (J, y)$ to $f'(x)$, determines a functor $X \to Y$. On which basic fact of elementary Calculus does this rely?

1.2 Some special objects and arrows

We call an arrow $f : A \to B$ *mono* (or a monomorphism, or monomorphic) in a category $\mathcal{C}$, if for any other object C and for any pair of arrows $g, h : C \to A$, $fg = fh$ implies $g = h$.

In Set, f is mono iff f is an injective function. The same is true for Grp, Grph, Rng, Preord, Pos,...

We call an arrow $f : A \to B$ *epi* (epimorphism, epimorphic) if for any pair $g, h : B \to C$, $gf = hf$ implies $g = h$.

The definition of epi is "dual" to the definition of mono. That is, f is epi in the category $\mathcal{C}$ if and only if f is mono in $\mathcal{C}^{\mathrm{op}}$, and vice versa. In general, given a property P of an object, arrow, diagram,... we can associate with P the dual property P^{op}: the object or arrow has property P^{op} in $\mathcal{C}$ iff it has P in $\mathcal{C}^{\mathrm{op}}$.

The *duality principle*, a very important, albeit trivial, principle in category theory, says that any valid statement about categories, involving the properties $P_1, \ldots, P_n$ implies the "dualized" statement (where direction of arrows is reversed) with the P_i replaced by P_i^{op}.

Example. If gf is mono, then f is mono. From this, "by duality", if fg is epi, then f is epi.

Exercise 10 Prove these statements.

In Set, f is epi iff f is a surjective function. This holds (less trivially!) also for Grp, but not for Mon, the category of monoids and monoid homomorphisms:

Example. In Mon, the embedding $\mathbb{N} \to \mathbb{Z}$ is an epimorphism.

For, suppose $\mathbb{Z} \overset{f}{\underset{g}{\rightrightarrows}} (M, e, \star)$ two monoid homomorphisms which agree on the nonnegative integers. Then

$$f(-1) = f(-1) \star g(1) \star g(-1) = f(-1) \star f(1) \star g(-1) = g(-1)$$

so f and g agree on the whole of $\mathbb{Z}$.

We say a functor F *preserves* a property P if whenever an object or arrow (or...) has P, its F-image does so.

Now a functor does not in general preserve monos or epis: the example of Mon shows that the forgetful functor Mon $\to$ Set does not preserve epis.

An epi $f : A \to B$ is called *split* if there is $g : B \to A$ such that $fg = \mathrm{id}_B$ (other names: in this case g is called a *section* of f, and f a *retraction* of g).

Exercise 11 By duality, define what a split mono is. Prove that every functor preserves split epis and monos.

A morphism $f : A \to B$ is an *isomorphism* if there is $g : B \to A$ such that $fg = \mathrm{id}_B$ and $gf = \mathrm{id}_A$. We call g the *inverse* of f (and vice versa, of course); it is unique if it exists. We also write $g = f^{-1}$.

Every functor preserves isomorphisms.

Exercise 12 In Set, every arrow which is both epi and mono is an isomorphism. Not so in Mon, as we have seen. Here's another one: let CRng1 be the category of commutative rings with 1, and ring homomorphisms (preserving 1) as arrows. Show that the embedding $\mathbb{Z} \to \mathbb{Q}$ is epi in CRng1.

Exercise 13 i) If two of f, g and fg are iso, then so is the third;

ii) if f is epi and split mono, it is iso;

iii) if f is split epi and mono, f is iso.

A functor F *reflects* a property P if whenever the F-image of something (object, arrow,...) has P, then that something has.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if for every two objects A, B of $\mathcal{C}$, $F : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is a surjection. F is *faithful* if this map is always injective.

Exercise 14 A faithful functor reflects epis and monos.

An object X is called *terminal* if for any object Y there is exactly one morphism $Y \to X$ in the category. Dually, X is *initial* if for all Y there is exactly one $X \to Y$.

Exercise 15 A full and faithful functor reflects the property of being a terminal (or initial) object.

Exercise 16 If X and X' are two terminal objects, they are *isomorphic*, that is there exists an isomorphism between them. Same for initial objects.

Exercise 17 Let $\sim$ be a congruence on the category $\mathcal{C}$, as in example g). Show: if f and g are arrows $X \to Y$ with inverses f^{-1} and g^{-1} respectively, then $f \sim g$ iff $f^{-1} \sim g^{-1}$.

Exercise 18 Recall that a topological space is *normal* if every one-point subset is closed and for every pair A, B of disjoint closed subsets, there exist disjoint open subsets U, V with $A \subset U$, $B \subset V$. We denote by $\mathcal{N}$ the full subcategory of Top on the normal topological spaces.

a) Characterise the epimorphisms in $\mathcal{N}$. Hint: you may find it useful to invoke Urysohn's Lemma.

b) Show that for two morphisms $f, g : A \to B$ in $\mathcal{N}$ we have: $f = g$ if and only if for every morphism $h : B \to \mathbb{R}$, $hf = hg$ holds (this property of $\mathbb{R}$ in $\mathcal{N}$ is sometimes called a *coseparator*)

2 Natural transformations

2.1 The Yoneda lemma

A *natural transformation* between two functors $F, G : \mathcal{C} \to \mathcal{D}$ consists of a family of morphisms $(\mu_C : FC \to GC)_{C \in \mathcal{C}_0}$ indexed by the collection of objects of $\mathcal{C}$, satisfying the following requirement: for every morphism $f : C \to C'$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} FC \arrow[r, "\mu_C"] \arrow[d, "Ff"'] & GC \arrow[d, "Gf"] \\ FC' \arrow[r, "\mu_{C'}"'] & GC' \end{tikzcd}$$

commutes in $\mathcal{D}$ (the diagram above is called the naturality square). We say $\mu = (\mu_C)_{C \in \mathcal{C}_0} : F \Rightarrow G$ and we call μ_C the *component at* C of the natural transformation μ.

Given natural transformations $\mu : F \Rightarrow G$ and $\nu : G \Rightarrow H$ we have a natural transformation $\nu\mu = (\nu_C \mu_C)_C : F \Rightarrow H$, and with this composition there is a category $\mathcal{D}^{\mathcal{C}}$ with functors $F : \mathcal{C} \to \mathcal{D}$ as objects, and natural transformations as arrows.

One of the points of the naturality square condition in the definition of a natural transformation is given by the following proposition. Compare with the situation in Set: denoting the set of all functions from X to Y by Y^X, for any set Z there is a bijection between functions $Z \to Y^X$ and functions $Z \times X \to Y$ (Set is *cartesian closed*: see chapter 7).

Proposition 2.1 *For categories $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ there is a bijection:*

$$\mathrm{Cat}(\mathcal{E} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\sim} \mathrm{Cat}(\mathcal{E}, \mathcal{D}^{\mathcal{C}})$$

Proof. Given $F : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$ define for every object E of $\mathcal{E}$ the functor $F_E : \mathcal{C} \to \mathcal{D}$ by $F_E(C) = F(E, C)$; for $f : C \to C'$ let $F_E(f) = F(\mathrm{id}_E, f) : F_E(C) = F(E, C) \to F(E, C') = F_E(C')$

Given $g : E \to E'$ in $\mathcal{E}$, the family $(F(g, \mathrm{id}_C) : F_E(C) \to F_{E'}(C))_{C \in \mathcal{C}_0}$ is a natural transformation: $F_E \Rightarrow F_{E'}$. So we have a functor $F \mapsto F_{(-)} : \mathcal{E} \to \mathcal{D}^{\mathcal{C}}$.

Conversely, given a functor $G : \mathcal{E} \to \mathcal{D}^{\mathcal{C}}$ we define a functor $\tilde{G} : \mathcal{E} \times \mathcal{C} \to \mathcal{D}$ on objects by $\tilde{G}(E, C) = G(E)(C)$, and on arrows by $\tilde{G}(g, f) = G(g)_{C'} G(E)(f) = G(E')(f) G(g)_C$:

$$\begin{tikzcd} G(E)(C) = \tilde{G}(E, C) \arrow[r, "G(g)_C"] \arrow[d, "G(E)(f)"'] & \tilde{G}(E', C) = G(E')(C) \arrow[d, "G(E')(f)"] \\ \tilde{G}(E, C') \arrow[r, "G(g)_{C'}"'] & \tilde{G}(E', C') = G(E')(C') \end{tikzcd}$$

Exercise 19 Write out the details. Check that $\tilde{G}$ as just defined, is a functor, and that the two operations

$$\mathrm{Cat}(\mathcal{E}\times\mathcal{C},\mathcal{D}) \longleftrightarrow \mathrm{Cat}(\mathcal{E},\mathcal{D}^{\mathcal{C}})$$

are inverse to each other.

■

An important example of natural transformations arises from the functors $y_C : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (see example i) in the preceding chapter); defined on objects by $y_C(C') = \mathcal{C}(C', C)$ and on arrows $f : C'' \to C'$ so that $y_C(f)$ is composition with f: $\mathcal{C}(C', C) \to \mathcal{C}(C'', C)$.

Given $g : C_1 \to C_2$ there is a natural transformation

$$y_g : y_{C_1} \Rightarrow y_{C_2}$$

whose components are composition with g.

Exercise 20 Spell this out.

We have, in other words, a functor

$$Y = y_{(-)} : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

This functor is called the *Yoneda embedding.*

An embedding is a functor which is full and faithful and injective on objects. That Y is injective on objects is easy to see, because $\mathrm{id}_C \in y_C(C)$ for each object C, and id_C is in no other set $y_D(E)$; that Y is full and faithful follows from the following

Proposition 2.2 (Yoneda lemma) *For every object F of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and every object C of $\mathcal{C}$, there is a bijection $f_{C,F} : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, F) \to F(C)$. Moreover, this bijection is* natural *in C and F in the following sense: given $g : C' \to C$ in $\mathcal{C}$ and $\mu : F \Rightarrow F'$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, the diagram*

$$\begin{tikzcd}
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, F) \arrow[r, "f_{C,F}"] \arrow[d, "\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(g,\mu)"'] & F(C) \arrow[d, "\mu_{C'}F(g)=F'(g)\mu_C"] \\
\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_{C'}, F') \arrow[r, "f_{C',F'}"'] & F'(C')
\end{tikzcd}$$

commutes in Set.

Proof. For every object C' of $\mathcal{C}$, every element f of $y_C(C') = \mathcal{C}(C', C)$ is equal to $\mathrm{id}_C f$ which is $y_C(f)(\mathrm{id}_C)$.

If $\kappa = (\kappa_{C'} | C' \in \mathcal{C}_0)$ is a natural transformation: $y_C \Rightarrow F$ then, $\kappa_{C'}(f)$ must be equal to $F(f)(\kappa_C(\mathrm{id}_C))$. So κ is completely determined by $\kappa_C(\mathrm{id}_C) \in F(C)$

and conversely, any element of $F(C)$ determines a natural transformation $y_C \Rightarrow F$.

Given $g : C' \to C$ in $\mathcal{C}$ and $\mu : F \Rightarrow F'$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, the map $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(g, \mu)$ sends the natural transformation $\kappa = (\kappa_{C''} | C'' \in \mathcal{C}_0) : y_C \Rightarrow F$ to $\lambda = (\lambda_{C''} | C'' \in \mathcal{C}_0)$ where $\lambda_{C''} : y_{C'}(C'') \to F'(C'')$ is defined by

$$\lambda_{C''}(h : C'' \to C') = \mu_{C''}(\kappa_{C''}(gh))$$

Now

$$\begin{array}{rcl} f_{C',F'}(\lambda) & = & \lambda_{C'}(\mathrm{id}_{C'}) \\ & = & \mu_{C'}(\kappa_{C'}(g)) \\ & = & \mu_{C'}(F(g)(\kappa_C(\mathrm{id}_C))) \\ & = & (\mu_{C'} F(g))(f_{C,F}(\kappa)) \end{array}$$

which proves the naturality statement. ∎

Corollary 2.3 *The functor $Y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is full and faithful.*

Proof. Immediate by the Yoneda lemma, since

$$\mathcal{C}(C, C') = y_{C'}(C) \cong \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, y_{C'})$$

and this bijection is induced by Y. ∎

The use of the Yoneda lemma is often the following. One wants to prove that objects A and B of $\mathcal{C}$ are isomorphic. Suppose one can show that for every object X of $\mathcal{C}$ there is a bijection $f_X : \mathcal{C}(X, A) \to \mathcal{C}(X, B)$ which is natural in X; i.e. given $g : X' \to X$ in $\mathcal{C}$ one has that

$$\begin{tikzcd} \mathcal{C}(X,A) \arrow[r, "f_X"] \arrow[d, "\mathcal{C}(g,\mathrm{id}_A)"'] & \mathcal{C}(X,B) \arrow[d, "\mathcal{C}(g,\mathrm{id}_B)"] \\ \mathcal{C}(X',A) \arrow[r, "f_{X'}"'] & \mathcal{C}(X',B) \end{tikzcd}$$

commutes.

Then one can conclude that A and B are isomorphic in $\mathcal{C}$; for, from what one has just shown it follows that y_A and y_B are isomorphic objects in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$; that is, $Y(A)$ and $Y(B)$ are isomorphic. Since Y is full and faithful, A and B are isomorphic by the following exercise:

Exercise 21 Check: if $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, and $F(A)$ is isomorphic to $F(B)$ in $\mathcal{D}$, then A is isomorphic to B in $\mathcal{C}$.

Exercise 22 Suppose objects A and B are such that for every object X in $\mathcal{C}$ there is a bijection $f_X : \mathcal{C}(A, X) \to \mathcal{C}(B, X)$, naturally in a sense you define yourself. Conclude that A and B are isomorphic (hint: duality + the previous).

This argument can be carried further. Suppose one wants to show that two functors $F, G : \mathcal{C} \to \mathcal{D}$ are isomorphic as objects of $\mathcal{D}^{\mathcal{C}}$. Let's first spell out what this means:

Exercise 23 Show that F and G are isomorphic in $\mathcal{D}^{\mathcal{C}}$ if and only if there is a natural transformation $\mu : F \Rightarrow G$ such that all components μ_C are isomorphisms (in particular, if μ is such, the family $((\mu_C)^{-1} | C \in \mathcal{C}_0)$ is also a natural transformation $G \Rightarrow F$).

Now suppose one has for each $C \in \mathcal{C}_0$ and $D \in \mathcal{D}_0$ a bijection

$$\mathcal{D}(D, FC) \cong \mathcal{D}(D, GC)$$

natural in D and C. This means that the objects y_{FC} and y_{GC} of $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ are isomorphic, by isomorphisms which are natural in C. By full and faithfulness of Y, FC and GC are isomorphic in $\mathcal{D}$ by isomorphisms natural in C; which says exactly that F and G are isomorphic as objects of $\mathcal{D}^{\mathcal{C}}$.

2.2 Examples of natural transformations

a) Let M and N be two monoids, regarded as categories with one object as in chapter 1. A functor $F : M \to N$ is then just the same as a homomorphism of monoids. Given two such, say $F, G : M \to N$, a natural transformation $F \Rightarrow G$ is (given by) an element n of N such that $nF(x) = G(x)n$ for all $x \in M$;

b) Let P and Q two preorders, regarded as categories. A functor $P \to Q$ is a monotone function, and there exists a unique natural transformation between two such, $F \Rightarrow G$, exactly if $F(x) \leq G(x)$ for all $x \in P$.

Exercise 24 In fact, show that if $\mathcal{D}$ is a preorder and the category $\mathcal{C}$ is *small*, i.e. the classes $\mathcal{C}_0$ and $\mathcal{C}_1$ are sets, then the functor category $\mathcal{D}^{\mathcal{C}}$ is a preorder.

c) Let $U : \mathrm{Grp} \to \mathrm{Set}$ denote the forgetful functor, and $F : \mathrm{Set} \to \mathrm{Grp}$ the free functor (see chapter 1). There are natural transformations $\varepsilon : FU \Rightarrow \mathrm{id}_{\mathrm{Grp}}$ and $\eta : \mathrm{id}_{\mathrm{Set}} \Rightarrow UF$.

   Given a group G, ε_G takes the string $\sigma = g_1 \dots g_n$ to the product $g_1 \cdots g_n$ (here, the "formal inverses" g_i^{-1} are interpreted as the real inverses in G!).

   Given a set A, $\eta_A(a)$ is the singleton string a.

d) Let $i : \mathrm{Abgp} \to \mathrm{Grp}$ be the inclusion functor and $r : \mathrm{Grp} \to \mathrm{Abgp}$ the abelianization functor defined in example m) in chapter 1. There is $\varepsilon : ri \Rightarrow \mathrm{id}_{\mathrm{Abgp}}$ and $\eta : \mathrm{id}_{\mathrm{Grp}} \Rightarrow ir$.

   The components η_G of η are the quotient maps $G \to G/[G, G]$; the components of ε are isomorphisms.

e) There are at least two ways to associate a category to a set X: let $F(X)$ be the category with as objects the elements of X, and as only arrows identities (a category of the form $F(X)$ is called *discrete*; and $G(X)$ the category with the same objects but with exactly one arrow $f_{x,y} : x \to y$ for each pair (x, y) of elements of X (We might call $G(X)$ an *indiscrete category*).

Exercise 25 Check that F and G can be extended to functors: Set $\to$ Cat and describe the natural transformation $\mu : F \Rightarrow G$ which has, at each component, the identity function on objects.

f) Every class of arrows of a category $\mathcal{C}$ can be viewed as a natural transformation. Suppose $S \subseteq \mathcal{C}_1$. Let $F(S)$ be the discrete category on S as in the preceding example. There are the two functors $\mathrm{dom}, \mathrm{cod} : F(S) \to \mathcal{C}$, giving the domain and the codomain, respectively. For every $f \in S$ we have $f : \mathrm{dom}(f) \to \mathrm{cod}(f)$, and the family $(f|f \in S)$ defines a natural transformation: $\mathrm{dom} \Rightarrow \mathrm{cod}$.

g) Let A and B be sets. There are functors $(-)\times A : \mathrm{Set} \to \mathrm{Set}$ and $(-)\times B : \mathrm{Set} \to \mathrm{Set}$. Any function $f : A \to B$ gives a natural transformation $(-) \times A \Rightarrow (-) \times B$.

h) A category $\mathcal{C}$ is called a *groupoid* if every arrow of $\mathcal{C}$ is an isomorphism. Let $\mathcal{C}$ be a groupoid, and suppose we are given, for each object X of $\mathcal{C}$, an arrow μ_X in $\mathcal{C}$ with domain X.

Exercise 26 Show that there is a functor $F : \mathcal{C} \to \mathcal{C}$ in this case, which acts on objects by $F(X) = \mathrm{cod}(\mu_X)$, and that $\mu = (\mu_X|X \in \mathcal{C}_0)$ is a natural transformation: $\mathrm{id}_{\mathcal{C}} \Rightarrow F$.

i) Given categories $\mathcal{C}$, $\mathcal{D}$ and an object D of $\mathcal{D}$, there is the constant functor $\Delta_D : \mathcal{C} \to \mathcal{D}$ which assigns D to every object of $\mathcal{C}$ and id_D to every arrow of $\mathcal{C}$.

   Every arrow $f : D \to D'$ gives a natural transformation $\Delta_f : \Delta_D \Rightarrow \Delta_{D'}$ defined by $(\Delta_f)_C = f$ for each $C \in \mathcal{C}_0$.

j) Let $\mathcal{P}(X)$ denote the power set of a set X: the set of subsets of X. The powerset operation can be extended to a functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$. Given a function $f : X \to Y$ define $\mathcal{P}(f)$ by $\mathcal{P}(f)(A) = f[A]$, the image of $A \subseteq X$ under f.

   There is a natural transformation $\eta : \mathrm{id}_{\mathrm{Set}} \Rightarrow \mathcal{P}$ such that $\eta_X(x) = \{x\} \in \mathcal{P}(X)$ for each set X.

   There is also a natural transformation $\mu : \mathcal{P}\mathcal{P} \Rightarrow \mathcal{P}$. Given $A \in \mathcal{P}\mathcal{P}(X)$, so A is a set of subsets of X, we take its union $\bigcup(A)$ which is a subset of X. Put $\mu_X(A) = \bigcup(A)$.

2.3 Equivalence of categories; an example

As will become clear in the following chapters, equality between objects plays only a minor role in category theory. The most important categorical notions are only defined “up to isomorphism”. This is in accordance with mathematical practice and with common sense: just renaming all elements of a group does not really give you another group.

We have already seen one example of this: the property of being a terminal object defines an object up to isomorphism. That is, any two terminal objects are isomorphic. There is, in the language of categories, no way of distinguishing between two isomorphic objects, so this is as far as we can get.

However, once we also consider functor categories, it turns out that there is another relation of “sameness” between categories, weaker than isomorphism of categories, and yet preserving all “good” categorical properties. Isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ requires the existence of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $FG = \mathrm{id}_{\mathcal{D}}$ and $GF = \mathrm{id}_{\mathcal{C}}$; but bearing in mind that we can't really say meaningful things about equality between objects, we may relax the requirement by just asking that FG is *isomorphic* to $\mathrm{id}_{\mathcal{D}}$ in the functor category $\mathcal{D}^{\mathcal{D}}$ (and the same for GF); doing this we arrive at the notion of *equivalence of categories*, which is generally regarded as the proper notion of sameness.

So two categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ and natural transformations $\mu : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\nu : \mathrm{id}_{\mathcal{D}} \Rightarrow FG$ whose components are all isomorphisms. F and G are called *pseudo inverses* of each other. A functor which has a pseudo inverse is also called an *equivalence of categories.*

As a simple example of an equivalence of categories, take a preorder P. Let Q be the quotient of P by the equivalence relation which contains the pair (x, y) iff both $x \leq y$ and $y \leq x$ in P. Let $\pi : P \to Q$ be the quotient map. Regarding P and Q as categories, π is a functor, and in fact an equivalence of categories, though not in general an isomorphism.

Exercise 27 Work this out.

Exercise 28 Show that a category is equivalent to a discrete category if and only if it is a groupoid and a preorder.

In this section I want to give an important example of an equivalence of categories: the so-called “Lindenbaum-Tarski duality between Set and Complete Atomic Boolean Algebras”. A duality between categories $\mathcal{C}$ and $\mathcal{D}$ is an equivalence between $\mathcal{C}^{\mathrm{op}}$ and $\mathcal{D}$ (equivalently, between $\mathcal{C}$ and $\mathcal{D}^{\mathrm{op}}$).

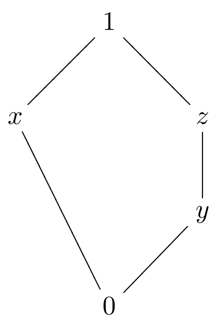
We need some definitions. A *lattice* is a partially ordered set in which every two elements x, y have a least upper bound (or join) $x \vee y$ and a greatest lower bound (or meet) $x \wedge y$; moreover, there exist a least element 0 and a greatest element 1.

Such a lattice is called a *Boolean algebra* if every element x has a *complement* $\neg x$, that is, satisfying $x \vee \neg x = 1$ and $x \wedge \neg x = 0$; and the lattice is *distributive*, which means that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all x, y, z.

In a Boolean algebra, complements are unique, for if both y and z are complements of x, then

$$y = y \wedge 1 = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = 0 \vee (y \wedge z) = y \wedge z$$

so $y \leq z$; similarly, $z \leq y$ so $y = z$. This is a non-example:



It is a lattice, and every element has a complement, but it is not distributive (check!).

A Boolean algebra B is *complete* if every subset A of B has a least upper bound $\bigvee A$ and a greatest lower bound $\bigwedge A$.

An *atom* in a Boolean algebra is an element x such that $0 < x$ but for no y we have $0 < y < x$. A Boolean algebra is atomic if every x is the join of the atoms below it:

$$x = \bigvee \{a | a \leq x \text{ and } a \text{ is an atom}\}$$

The category CABool is defined as follows: the objects are complete atomic Boolean algebras, and the arrows are complete homomorphisms, that is: $f : B \to C$ is a complete homomorphism if for every $A \subseteq B$,

$$f(\bigvee A) = \bigvee \{f(a) | a \in A\} \text{ and } f(\bigwedge A) = \bigwedge \{f(a) | a \in A\}$$

Exercise 29 Show that $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$. Conclude that a complete homomorphism preserves 1, 0 and complements.

Exercise 30 Show that $\bigwedge A = \neg \bigvee \{\neg a | a \in A\}$ and conclude that if a function preserves all $\bigvee$'s, 1 and complements, it is a complete homomorphism.

Theorem 2.4 *The category* CABool *is equivalent to* $\mathrm{Set}^{\mathrm{op}}$.

Proof. For every set X, $\mathcal{P}(X)$ is a complete atomic Boolean algebra and if $f : Y \to X$ is a function, then $f^{-1} : \mathcal{P}(X) \to \mathcal{P}(Y)$ which takes, for each subset of X, its inverse image under f, is a complete homomorphism. So this defines a functor $F : \mathrm{Set}^{\mathrm{op}} \to \mathrm{CABool}$.

Conversely, given a complete atomic Boolean algebra B, let $G(B)$ be the set of atoms of B. Given a complete homomorphism $g : B \to C$ we have a function $G(g) : G(C) \to G(B)$ defined by: $G(g)(c)$ is the unique $b \in G(B)$ such that $c \leq g(b)$. This is well-defined: first, there is an atom b with $c \leq g(b)$ because $1_B = \bigvee G(B)$ (B is atomic), so $1_C = g(1_B) = \bigvee\{g(b)|b \text{ is an atom}\}$ and:

Exercise 31 Prove: if c is an atom and $c \leq \bigvee A$, then there is $a \in A$ with $c \leq a$ (hint: prove for all a, b that $a \wedge b = 0 \Leftrightarrow a \leq \neg b$, and prove for a, c with c atom: $c \not\leq a \Leftrightarrow a \leq \neg c$).

Secondly, the atom b is unique since $c \leq g(b)$ and $c \leq g(b')$ means $c \leq g(b) \wedge g(b') = g(b \wedge b') = g(0) = 0$.

So we have a functor $G : \mathrm{CABool} \to \mathrm{Set}^{\mathrm{op}}$.

Now the atoms of the Boolean algebra $\mathcal{P}(X)$ are exactly the singleton subsets of X, so $GF(X) = \{\{x\}|x \in X\}$ which is clearly isomorphic to X. On the other hand, $FG(B) = \mathcal{P}(\{b \in B \,|\, b \text{ is an atom}\})$. There is a map from $FG(B)$ to B which sends each set of atoms to its least upper bound in B, and this map is an isomorphism in CABool. ∎

Exercise 32 Prove the last statement: that the map from $FG(B)$ to B, defined in the last paragraph of the proof, is an isomorphism.

Exercise 33 Prove that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is full and faithful, and *essentially surjective on objects*, that means: for any $D \in \mathcal{D}_0$ there is $C \in \mathcal{C}_0$ such that $F(C)$ is isomorphic to D.

3 (Co)cones and (Co)limits

3.1 Limits

Given a functor $F : \mathcal{C} \to \mathcal{D}$, a *cone* for F consists of an object D of $\mathcal{D}$ together with a natural transformation $\mu : \Delta_D \Rightarrow F$ (Δ_D is the constant functor with value D). In other words, we have a family $(\mu_C : D \to F(C) | C \in \mathcal{C}_0)$, and the naturality requirement in this case means that for every arrow $f : C \to C'$ in $\mathcal{C}$,

$$\begin{tikzcd} & D \arrow[dl, "\mu_C"'] \arrow[dr, "\mu_{C'}"] & \\ F(C) \arrow[rr, "F(f)"'] & & F(C') \end{tikzcd}$$

commutes in $\mathcal{D}$ (this diagram explains, I hope, the name "cone"). Let us denote the cone by (D, μ). D is called the *vertex* of the cone.

A *map of cones* $(D, \mu) \to (D', \mu')$ is a map $g : D \to D'$ such that $\mu'_C g = \mu_C$ for all $C \in \mathcal{C}_0$.

Clearly, there is a category Cone(F) which has as objects the cones for F and as morphisms maps of cones.

A *limiting cone* for F is a terminal object in Cone(F). Since terminal objects are unique up to isomorphism, as we have seen, any two limiting cones are isomorphic in Cone(F) and in particular, their vertices are isomorphic in $\mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is also called a *diagram* in $\mathcal{D}$ *of type* $\mathcal{C}$, and $\mathcal{C}$ is the *index category* of the diagram.

Let us see what it means to be a limiting cone, in some simple, important cases.

i) A limiting cone for the unique functor $! : \mathbf{0} \to \mathcal{D}$ ($\mathbf{0}$ is the empty category) "is" a terminal object in $\mathcal{D}$. For every object D of $\mathcal{D}$ determines, together with the empty family, a cone for !, and a map of cones is just an arrow in $\mathcal{D}$. So Cone(!) is isomorphic to $\mathcal{D}$.

ii) Let $\mathbf{2}$ be the discrete category with two objects x, y. A functor $\mathbf{2} \to \mathcal{D}$ is just a pair $\langle A, B \rangle$ of objects of $\mathcal{D}$, and a cone for this functor consists of an object C of $\mathcal{D}$ and two maps $\begin{tikzcd} C \arrow[r, "\mu_A"] \arrow[dr, "\mu_B"'] & A \\ & B \end{tikzcd}$ since there are no nontrivial arrows in $\mathbf{2}$.

$(C, (\mu_A, \mu_B))$ is a limiting cone for $\langle A, B \rangle$ iff the following holds: for any object D and arrows $f : D \to A$, $g : D \to B$, there is a unique arrow

$h : D \to C$ such that

$$\begin{tikzcd} & C \arrow[d, "\mu_A"] \arrow[ddr, "\mu_B"] & \\ D \arrow[ur, dashed, "h"] \arrow[r, "f"] \arrow[drr, "g"'] & A & \\ & & B \end{tikzcd}$$

commutes. In other words, there is, for any D, a 1-1 correspondence between maps $D \to C$ and pairs of maps $\begin{tikzcd} & D \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$ This is the property of a *product*; a limiting cone for $\langle A, B\rangle$ is therefore called a product cone, and usually denoted:

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

The arrows π_A and π_B are called *projections*.

iii) Let $\hat{\mathbf{2}}$ denote the category $x \overset{a}{\underset{b}{\rightrightarrows}} y$. A functor $\hat{\mathbf{2}} \to \mathcal{D}$ is the same thing as a parallel pair of arrows $A \overset{f}{\underset{g}{\rightrightarrows}} B$ in $\mathcal{D}$; I write $\langle f, g\rangle$ for this functor. A cone for $\langle f, g\rangle$ is:

$$\begin{tikzcd} & D \arrow[dl, "\mu_A"'] \arrow[dr, "\mu_B"] & \\ A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & B \end{tikzcd}$$

But $\mu_B = f\mu_A = g\mu_A$ is already defined from μ_A, so giving a cone is the same as giving a map $\mu_A : D \to A$ such that $f\mu_A = g\mu_A$. Such a cone is limiting iff for any other map $h : C \to A$ with $fh = gh$, there is a unique $k : C \to D$ such that $h = \mu_A k$.

We call μ_A, if it is limiting, an *equalizer* of the pair f, g, and the diagram $D \xrightarrow{\mu_A} A \overset{f}{\underset{g}{\rightrightarrows}} B$ an equalizer diagram.

In Sets, an equalizer of f, g is isomorphic (as a cone) to the inclusion of $\{a \in A | f(a) = g(a)\}$ into A. In categorical interpretations of logical systems (see chapters 4 and 7), equalizers are used to interpret *equality between terms*.

Exercise 34 Show that every equalizer is a monomorphism.

Exercise 35 If $E \xrightarrow{e} X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is an equalizer diagram, show that e is an isomorphism if and only if $f = g$.

Exercise 36 Show that in Set, every monomorphism fits into an equalizer diagram.

iv) Let J denote the category
$$\begin{tikzcd} & y \arrow[d, "b"] \\ x \arrow[r, "a"'] & z \end{tikzcd}$$
A functor $F : J \to \mathcal{D}$ is specified by giving two arrows in $\mathcal{D}$ with the same codomain, say $f : X \to Z$, $g : Y \to Z$. A limit for such a functor is given by an object W together with projections
$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \\ X & \end{tikzcd}$$
satisfying $fp_X = gp_Y$, and such that, given any other pair of arrows:
$$\begin{tikzcd} V \arrow[r, "r"] \arrow[d, "s"'] & Y \\ X & \end{tikzcd}$$
with $gr = fs$, there is a unique arrow $V \to W$ such that
$$\begin{tikzcd} V \arrow[dr] \arrow[drr, "r"] \arrow[ddr, "s"'] & & \\ & W \arrow[r, "p_Y"'] \arrow[d, "p_X"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Z \end{tikzcd}$$
commutes.

The diagram
$$\begin{tikzcd} W \arrow[r, "p_Y"] \arrow[d, "p_X"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"'] & Z \end{tikzcd}$$
is called a *pullback diagram.* In Set, the pullback cone for f, g is isomorphic to
$$\{(x, y) \in X \times Y | f(x) = g(y)\}$$
with the obvious projections.

We say that a category $\mathcal{D}$ *has binary products* (equalizers, pullbacks) iff every functor $\mathbf{2} \to \mathcal{D}$ ($\hat{\mathbf{2}} \to \mathcal{D}$, $J \to \mathcal{D}$, respectively) has a limiting cone. Some dependencies hold in this context:

Proposition 3.1 *If a category $\mathcal{D}$ has a terminal object and pullbacks, it has binary products and equalizers.*
If $\mathcal{D}$ has binary products and equalizers, it has pullbacks.

Proof. Let 1 be the terminal object in $\mathcal{D}$; given objects X and Y, if
$$\begin{tikzcd} C \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \arrow[d] \\ Y \arrow[r] & 1 \end{tikzcd}$$
is a pullback diagram, then
$$\begin{tikzcd} C \arrow[r, "p_X"] \arrow[d, "p_Y"'] & X \\ Y & \end{tikzcd}$$
is a product cone.

Given a product cone
$$\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \\ B & \end{tikzcd}$$
and maps
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & A \\ B & \end{tikzcd}$$
we write $X \overset{\langle f,g \rangle}{\to} A \times B$ for the unique factorization through the product. Write also $\delta : Y \to Y \times Y$ for $\langle \mathrm{id}_Y, \mathrm{id}_Y \rangle$.

Now given $f, g : X \to Y$, if

$$\begin{tikzcd} E \arrow[r, "e"] \arrow[d] & X \arrow[d, "\langle f,g \rangle"] \\ Y \arrow[r, "\delta"'] & Y \times Y \end{tikzcd}$$

is a pullback diagram, then $E \xrightarrow{e} X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is an equalizer diagram. This proves the first statement.

As for the second: given
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
let $E \xrightarrow{e} X \times Y \underset{g\pi_Y}{\overset{f\pi_X}{\rightrightarrows}} Z$ be an equalizer; then

$$\begin{tikzcd} E \arrow[r, "\pi_X e"] \arrow[d, "\pi_Y e"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

is a pullback diagram. ∎

Exercise 37 Let

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[d, "f"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

a pullback diagram with f mono. Show that a is also mono. Also, if f is iso (an isomorphism), so is a.

Exercise 38 Given two commuting squares:

$$\begin{tikzcd} A \arrow[r, "b"] \arrow[d, "a"'] & B \arrow[r, "c"] \arrow[d, "f"'] & C \arrow[d, "d"'] \\ X \arrow[r, "g"'] & Y \arrow[r, "h"'] & Z \end{tikzcd}$$

a) if both squares are pullback squares, then so is the composite square
$$\begin{tikzcd} A \arrow[r, "cb"] \arrow[d, "a"'] & C \arrow[d, "d"] \\ X \arrow[r, "hg"'] & Z \end{tikzcd}$$

b) If the right hand square and the composite square are pullbacks, then so is the left hand square.

Exercise 39 $f : A \to B$ is a monomorphism if and only if

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\mathrm{id}_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback diagram.

A monomorphism $f : A \to B$ which fits into an equalizer diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

is called a *regular mono*.

Exercise 40 If

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "b"'] & X \arrow[d, "g"] \\ B \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback and g is regular mono, so is b.

Exercise 41 If f is regular mono and epi, f is iso. Every split mono is regular.

Exercise 42 Give an example of a category in which not every mono is regular.

Exercise 43 In Grp, every mono is regular [This is not so easy].

Exercise 44 Characterize the regular monos in Pos.

Exercise 45 If a category $\mathcal{D}$ has binary products and a terminal object, it has *all finite products*, i.e. limiting cones for every functor into $\mathcal{D}$ from a finite discrete category.

Exercise 46 Suppose $\mathcal{C}$ has binary products and suppose for every ordered pair (A, B) of objects of $\mathcal{C}$ a product cone $\begin{tikzcd} A \times B \arrow[r, "\pi_A"] \arrow[d, "\pi_B"'] & A \\ B \end{tikzcd}$ has been chosen.

a) Show that there is a functor: $\mathcal{C} \times \mathcal{C} \xrightarrow{- \times -} \mathcal{C}$ (the product functor) which sends each pair (A, B) of objects to $A \times B$ and each pair of arrows $(f : A \to A', g : B \to B')$ to $f \times g = \langle f\pi_A, g\pi_B \rangle$.

b) From a), there are functors:

$$\mathcal{C} \times \mathcal{C} \times \mathcal{C} \overset{(- \times -) \times -}{\underset{- \times (- \times -)}{\rightrightarrows}} \mathcal{C}$$

sending (A, B, C) to $\begin{array}{c} (A \times B) \times C \\ A \times (B \times C) \end{array}$ Show that there is a natural transformation $a = (a_{A,B,C} | A, B, C \in \mathcal{C}_0)$ from $(- \times -) \times -$ to $- \times (- \times -)$

such that for any four objects A, B, C, D of $\mathcal{C}$:

$$\begin{tikzcd}
((A \times B) \times C) \times D \arrow[rr, "a_{A\times B,C,D}"] \arrow[d, "a_{A,B,C}\times \mathrm{id}_D"'] & & (A \times B) \times (C \times D) \arrow[d, "a_{A,B,C\times D}"] \\
(A \times (B \times C)) \times D \arrow[dr, "a_{A,B\times C,D}"'] & & A \times (B \times (C \times D)) \\
& A \times ((B \times C) \times D) \arrow[ur, "\mathrm{id}_A \times a_{B,C,D}"'] &
\end{tikzcd}$$

commutes (This diagram is called "MacLane's pentagon").

A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits of type* $\mathcal{E}$ if for all functors $M : \mathcal{E} \to \mathcal{C}$, if (D, μ) is a limiting cone for M in $\mathcal{C}$, then $(FD, F\mu = (F(\mu_E)|E \in \mathcal{E}_0))$ is a limiting cone for FM in $\mathcal{D}$.

So, a functor $F : \mathcal{C} \to \mathcal{D}$ preserves binary products if for every product diagram $\begin{tikzcd} A \times B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] & B \\ A & \end{tikzcd}$ its F-image $\begin{tikzcd} F(A \times B) \arrow[r, "F(\pi_B)"] \arrow[d, "F(\pi_A)"'] & F(B) \\ F(A) & \end{tikzcd}$ is again a product diagram. Similarly for equalizers and pullbacks.

Some more terminology: F is said to *preserve all finite limits* if it preserves limits of type $\mathcal{E}$ for every finite $\mathcal{E}$. A category which has all finite limits is called *lex* (*left exact*), *cartesian* or *finitely complete*.

Exercise 47 If a category $\mathcal{C}$ has equalizers, it has *all finite equalizers*: for every category $\mathcal{E}$ of the form

$$\begin{tikzcd} X \arrow[r, shift left=2, "f_1"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2, "f_n"'] & Y \end{tikzcd}$$

every functor $\mathcal{E} \to \mathcal{C}$ has a limiting cone.

Exercise 48 Suppose $F : \mathcal{C} \to \mathcal{D}$ preserves equalizers (and $\mathcal{C}$ has equalizers) and reflects isomorphisms. Then F is faithful.

Exercise 49 Let $\mathcal{C}$ be a category with finite limits. Show that for every object C of $\mathcal{C}$, the slice category $\mathcal{C}/C$ (example j) of 1.1) has binary products: the vertex of a product diagram for two objects $D \to C$, $D' \to C$ is $D'' \to C$ where

$$\begin{tikzcd} D'' \arrow[r] \arrow[d] & D \arrow[d] \\ D' \arrow[r] & C \end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

3.2 Limits by products and equalizers

In Set, every small diagram has a limit; given a functor $F : \mathcal{E} \to \text{Set}$ with $\mathcal{E}$ small, there is a limiting cone for F in Set with vertex

$$\{(x_E)_{E\in\mathcal{E}_0} \in \prod_{E\in\mathcal{E}_0} F(E) \,|\, \forall E \xrightarrow{f} E' \in \mathcal{E}_1(F(f)(x_E) = x_{E'})\}$$

So in Set, limits are equationally defined subsets of suitable products. This holds in any category:

Proposition 3.2 *Suppose $\mathcal{C}$ has all small products (including the empty product, i.e. a terminal object 1) and equalizers; then $\mathcal{C}$ has all small limits.*

Proof. Given a set I and an I-indexed family of objects $(A_i|i \in I)$ of $\mathcal{C}$, we denote the product by $\prod_{i\in I} A_i$ and projections by $\pi_i : \prod_{i\in I} A_i \to A_i$; an arrow $f : X \to \prod_{i\in I} A_i$ which is determined by the compositions $f_i = \pi_i f : X \to A_i$, is also denoted $(f_i|i \in I)$.

Now given $\mathcal{E} \to \mathcal{C}$ with $\mathcal{E}_0$ and $\mathcal{E}_1$ sets, we construct

$$E \xrightarrow{e} \prod_{i\in\mathcal{E}_0} F(i) \overset{(\pi_{\mathrm{cod}(u)}|u\in\mathcal{E}_1)}{\underset{(F(u)\pi_{\mathrm{dom}(u)}|u\in\mathcal{E}_1)}{\rightrightarrows}} \prod_{u\in\mathcal{E}_1} F(\mathrm{cod}(u))$$

in $\mathcal{C}$ as an equalizer diagram. The family $(\mu_i = \pi_i e : E \to F(i)|i \in \mathcal{E}_0)$ is a natural transformation $\Delta_E \Rightarrow F$ because, given an arrow $u \in \mathcal{E}_1$, say $u : i \to j$, we have that

$$\begin{tikzcd} & E \arrow[dl, "\pi_i e"'] \arrow[dr, "\pi_j e"] & \\ F(i) \arrow[rr, "F(u)"'] & & F(j) \end{tikzcd}$$

commutes since $F(u)\pi_i e = F(u)\pi_{\mathrm{dom}(u)} e = \pi_{\mathrm{cod}(u)} e = \pi_j e$.

So (E, μ) is a cone for F, but every other cone (D, ν) for F gives a map $d : D \to \prod_{i\in\mathcal{E}_0} F(i)$ equalizing the two horizontal arrows; so factors uniquely through E. ∎

Exercise 50 Check that "small" in the statement of the proposition, can be replaced by "finite": if $\mathcal{C}$ has all finite products and equalizers, $\mathcal{C}$ is finitely complete.

Exercise 51 Show that if $\mathcal{C}$ is complete, then $F : \mathcal{C} \to \mathcal{D}$ preserves all limits if F preserves products and equalizers. This no longer holds if $\mathcal{C}$ is not complete! That is, F may preserve all products and equalizers which exist in $\mathcal{C}$, yet not preserve all limits which exist in $\mathcal{C}$.

3.3 Complete Categories

A category is called *complete* if it has limits of type $\mathcal{E}$ for all small $\mathcal{E}$.

In general, limits over large (i.e. not small) diagrams need not exist. For example in Set, there is a limiting cone for the identity functor Set $\to$ Set (its vertex is the empty set), but not for the constant functor $\Delta_A : \mathcal{C} \to \mathrm{Set}$ if $\mathcal{C}$ is a large discrete category and A has more than one element.

The categories Set, Top, Pos, Mon, Grp, Grph, Rng,... are all complete. For instance in Top, the product of a set $(X_i \mid i \in I)$ of topological spaces is the set $\prod_{i\in I} X_i$ with the product topology; the equalizer of two continuous maps $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the inclusion $X' \subseteq X$ where $X' = \{x \in X \mid f(x) = g(x)\}$ with the subspace topology from X.

Exercise 52 What are monomorphisms in Top? Is every mono regular in Top?

The category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is also complete, and limits are "computed pointwise". That is, let $F : \mathcal{D} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ be a diagram in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. For every $C \in \mathcal{C}_0$ there is a functor $F_C : \mathcal{D} \to \mathrm{Set}$, given by $F_C(D) = F(D)(C)$ and for $f : D \to D'$ in $\mathcal{D}$, $F_C(f) = F(f)_C : F(D)(C) \to F(D')(C)$.

Since Set is complete, every F_C has a limiting cone (X_C, μ_C) in Set. Now if $C' \xrightarrow{g} C$ is a morphism in $\mathcal{C}$, the collection of arrows

$$\{X_C \xrightarrow{(\mu_C)_D} F(D)(C) \xrightarrow{F(D)(g)} F(D)(C') = F_{C'}(D) \mid D \in \mathcal{D}_0\}$$

is a cone for $F_{C'}$ with vertex X_C, since for any $f : D \to D'$ we have $F(f)_{C'} \circ F(D)(g) \circ (\mu_C)_D = F(D')(g) \circ F(f)_C \circ (\mu_C)_D$ (by naturality of $F(f)$) $= F(D')(g) \circ (\mu_C)_{D'}$ (because (X_C, μ_C) is a cone).

Because $(X_{C'}, \mu_{C'})$ is a limiting cone for $F_{C'}$, there is a unique arrow $X_g : X_C \to X_{C'}$ in Set such that $F(D)(g) \circ (\mu_C)_D = (\mu_{C'})_D \circ X_g$ for all $D \in \mathcal{D}_0$. By the uniqueness of these arrows, we have an object X of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, and arrows $\nu_D : X \to F(D)$ for all $D \in \mathcal{D}_0$, and the pair (X, ν) is a limiting cone for F in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Exercise 53 Check the remaining details.

It is a consequence of the Yoneda lemma that the Yoneda embedding $Y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. For, let $F : \mathcal{D} \to \mathcal{C}$ be a diagram with limiting cone (E, ν) and let (X, δ) be a limiting cone for the composition $Y \circ F : \mathcal{D} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. By the Yoneda lemma, $X(C)$ is in natural 1-1 correspondence with the set of arrows $Y(C) \to X$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$; which by the fact that (X, δ) is a limiting cone, is in natural 1-1 correspondence with cones $(Y(C), \mu)$ for $Y \circ F$ with vertex $Y(C)$; since Y is full and faithful every such cone comes from a unique cone (C, μ') for F in $\mathcal{C}$, hence from a unique map $C \to E$ in $\mathcal{C}$.

So, $X(C)$ is naturally isomorphic to $\mathcal{C}(C, E)$ whence X is isomorphic to $Y(E)$, by an isomorphism which transforms δ into $Y \circ \nu = (Y(\nu_C) \mid C \in \mathcal{C}_0)$.

To finish this section a little theorem by Peter Freyd which says that every small, complete category is a complete preorder:

Proposition 3.3 *Suppose $\mathcal{C}$ is small and complete. Then $\mathcal{C}$ is a preorder.*

Proof. If not, there are objects A, B in $\mathcal{C}$ such that there are two distinct maps $f, g : A \to B$. Since $\mathcal{C}_1$ is a set and $\mathcal{C}$ complete, the product $\prod_{h\in\mathcal{C}_1} B$ exists. Arrows $k : A \to \prod_{h\in\mathcal{C}_1} B$ are in 1-1 correspondence with families of arrows $(k_h : A \to B \mid h \in \mathcal{C}_1)$. For every subset $X \subseteq \mathcal{C}_1$ define such a family by:

$$k_h = \begin{cases} f & \text{if } h \in X \\ g & \text{else} \end{cases}$$

This gives an injective function from $2^{\mathcal{C}_1}$ into $\mathcal{C}(A, \prod_{h\in\mathcal{C}_1} B)$ hence into $\mathcal{C}_1$, contradicting Cantor's theorem in set theory. ∎

3.4 Colimits

The dual notion of limit is colimit. Given a functor $F : \mathcal{E} \to \mathcal{C}$ there is clearly a functor $F^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ which does "the same" as F. We say that a *colimiting cocone* for F is a limiting cone for F^{op}.

So: a cocone for $F : \mathcal{E} \to \mathcal{C}$ is a pair (ν, D) where $\nu : F \Rightarrow \Delta_D$ and a colimiting cocone is an initial object in the category Cocone(F).

Examples

i) a colimiting cocone for $! : \mathbf{0} \to \mathcal{C}$ "is" an initial object of $\mathcal{C}$

ii) a colimiting cocone for $\langle A, B\rangle : \mathbf{2} \to \mathcal{C}$ is a *coproduct* of A and B in $\mathcal{C}$: usually denoted $A + B$ or $A \sqcup B$; there are *coprojections* or *coproduct inclusions*

$$\begin{tikzcd} A \arrow[dr, "\nu_A"] & \\ B \arrow[r, "\nu_B"'] & A \sqcup B \end{tikzcd}$$

with the property that, given any pair of arrows $A \xrightarrow{f} C$, $B \xrightarrow{g} C$ there is a unique map $\begin{bmatrix} f \\ g \end{bmatrix} : A \sqcup B \to C$ such that $f = \begin{bmatrix} f \\ g \end{bmatrix} \nu_A$ and $g = \begin{bmatrix} f \\ g \end{bmatrix} \nu_B$

iii) a colimiting cocone for $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$ (as functor $\hat{\mathbf{2}} \to \mathcal{C}$) is given by a map $B \xrightarrow{c} C$ satisfying $cf = cg$, and such that for any $B \xrightarrow{h} D$ with $hf = hg$ there is a unique $C \xrightarrow{h'} D$ with $h = h'c$. c is called a *coequalizer* for f and g; the diagram $\begin{tikzcd} A \arrow[r, shift left] \arrow[r, shift right] & B \arrow[r] & C \end{tikzcd}$ a coequalizer diagram.

Exercise 54 Is the terminology "coproduct inclusions" correct? That is, it suggests they are monos. Is this always the case?

Formulate a condition on A and B which implies that ν_A and ν_B are monic.

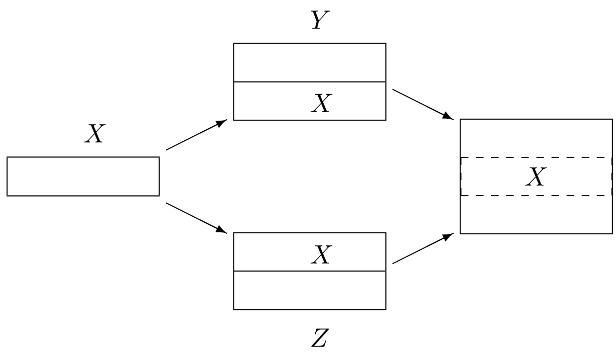
In Set, the coproduct of X and Y is the disjoint union $(\{0\} \times X) \cup (\{1\} \times Y)$ of X and Y. The coequalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ is the quotient map $Y \to Y/\sim$ where $\sim$ is the equivalence relation generated by

$$y \sim y' \text{ iff there is } x \in X \text{ with } f(x) = y \text{ and } g(x) = y'$$

The dual notion of pullback is *pushout.* A pushout diagram is a colimiting cocone for a functor $\Gamma \to \mathcal{C}$ where Γ is the category
$$\begin{tikzcd} x \arrow[r] \arrow[d] & y \\ z & \end{tikzcd}$$
. Such a diagram is a square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "a"] \\ Z \arrow[r, "b"'] & P \end{tikzcd}$$

which commutes and such that, given
$$\begin{tikzcd} Y \arrow[dr, "\alpha"] & \\ Z \arrow[r, "\beta"'] & Q \end{tikzcd}$$
with $\alpha f = \beta g$, there is a unique $P \xrightarrow{p} Q$ with $\alpha = pa$ and $\beta = pb$. In Set, the pushout of $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$ is the coproduct $Y \sqcup Z$ where the two images of X are identified:



Exercise 55 Give yourself, in terms of $X \xrightarrow{f} Y$ and $X \xrightarrow{g} Z$, a formal definition of a relation R on $Y \sqcup Z$ such that the pushout of f and g is $Y \sqcup Z/\sim$, $\sim$ being the equivalence relation generated by R.

One can now dualize every result and exercise from the section on limits:

Exercise 56 f is epi if and only if

$$\begin{tikzcd} \cdot \arrow[r, "f"] \arrow[d, "f"'] & \cdot \arrow[d, "\mathrm{id}"] \\ \cdot \arrow[r, "\mathrm{id}"'] & \cdot \end{tikzcd}$$

is a pushout diagram.

Exercise 57 Every coequalizer is an epimorphism; if e is a coequalizer of f and g, then e is iso iff $f = g$

Exercise 58 If $\mathcal{C}$ has an initial object and pushouts, $\mathcal{C}$ has binary coproducts and coequalizers; if $\mathcal{C}$ has binary coproducts and coequalizers, $\mathcal{C}$ has pushouts.

Exercise 59 If $\begin{tikzcd} \cdot \arrow[r] \arrow[d, "a"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ is a pushout diagram, then a epi implies f epi, and a regular epi (i.e. a coequalizer) implies f regular epi.

Exercise 60 The composition of two puhout squares is a pushout; if both the first square and the composition are pushouts, the second square is.

Exercise 61 If $\mathcal{C}$ has all small (finite) coproducts and coequalizers, $\mathcal{C}$ has all small (finite) colimits.

Some miscellaneous exercises:

Exercise 62 Call an arrow f a *stably regular epi* if whenever $\begin{tikzcd} \cdot \arrow[r] \arrow[d, "a"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$ is a pullback diagram, the arrow a is a regular epi. Show: in Pos, $X \xrightarrow{f} Y$ is a stably regular epi if and only if for all y, y' in Y:

$$y \leq y' \Leftrightarrow \exists x \in f^{-1}(y) \exists x' \in f^{-1}(y').x \leq x'$$

Show by an example that not every epi is stably regular in Pos.

Exercise 63 In Grp, every epi is regular.

Exercise 64 Characterize coproducts in Abgrp.

4 A little piece of categorical logic

One of the major achievements of category theory in mathematical logic and in computer science, has been a unified treatment of semantics for all kinds of logical systems and term calculi which are the basis for programming languages.

One can say that mathematical logic, seen as the study of classical first order logic, first started to be a real subject with the discovery, by Gödel, of the *completeness theorem* for set-theoretic interpretations: a sentence φ is provable if and only if φ is true in all possible interpretations. This unites the two approaches to logic: proof theory and model theory, makes logic accessible for mathematical methods and enables one to give nice and elegant proofs of proof theoretical properties by model theory (for example, the Beth and Craig definability and interpolation theorems).

However the completeness theorem needs generalization once one considers logics, such as intuitionistic logic (which does not admit the principle of excluded middle), minimal logic (which has no negation) or modal logic (where the logic has an extra operator, expressing "necessarily true"), for which the set-theoretic interpretation is not complete. One therefore comes with a general definition of "interpretation" in a category $\mathcal{C}$ of a logical system, which generalizes Tarski's truth definition: this will then be the special case of classical logic and the category Set.

In this chapter I treat, for reasons of space, only a fragment of first order logic: regular logic. On this fragment the valid statements of classical and intuitionistic logic coincide.

For an interpretation of a term calculus like the λ-calculus, which is of paramount importance in theoretical computer science, the reader is referred to chapter 7.

4.1 Regular categories and subobjects

Definition 4.1 *A category $\mathcal{C}$ is called* regular *if the following conditions hold:*

a) *$\mathcal{C}$ has all finite limits;*

b) *For every arrow f, if*

$$\begin{tikzcd} Z \arrow[r, "p_0"] \arrow[d, "p_1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback (then $Z \underset{p_1}{\overset{p_0}{\rightrightarrows}} X$ is called the kernel pair *of f), the coequalizer of p_0, p_1 exists;*

c) *Regular epimorphisms (coequalizers) are* stable under pullback*, that is: in* *a pullback square*

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "a"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$$

if f is regular epi, so is a.

Examples. In Set, as in Grp, Top, etc., the (underlying) set which is the vertex of the kernel pair of $f : X \to Y$ is $X_f = \{(x, x') | \, f(x) = f(x')\}$. The coequalizer of $X_f \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$ is (up to isomorphism) the map $X \to \mathrm{Im}(f)$ where $\mathrm{Im}(f)$ is the set-theoretic image of f as subset of Y.

These coequalizers exist in Set, Top, Grp, Pos. . . . Moreover, in Set and Grp every epi is regular, and (since epis in Set and Grp are just surjective functions) stable under pullback; hence Set and Grp are examples of regular categories.

Top is not regular! It satisfies the first two requirements of the definition, but not the third. One can prove that the functor $(-)\times S : \mathrm{Top} \to \mathrm{Top}$ preserves all quotient maps only if the space S is locally compact. Since every coequalizer is a quotient map, if S is not locally compact there will be pullbacks of form

$$\begin{tikzcd} X \times S \arrow[r, "f\times \mathrm{id}_S"] \arrow[d] & Y \times S \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with f regular epi, but $f \times \mathrm{id}_S$ not.

Exercise 65 This exercise shows that Pos is not regular either. Let X and Y be the partial orders $\{x \le y, y' \le z\}$ and $\{a \le b \le c\}$ respectively.

a) Prove that $f(x) = a$, $f(y) = f(y') = b$, $f(z) = c$ defines a regular epimorphism: $X \to Y$.

b) Let Z be $\{a \le c\} \subset Y$ and $W = f^{-1}(Z) \subset X$. Then

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

is a pullback, but $W \to Z$ is not the coequalizer of anything.

Proposition 4.2 *In a regular category, every arrow $f : X \to Y$ can be factored as $f = me : X \stackrel{e}{\to} E \stackrel{m}{\to} Y$ where e is regular epi and m is mono; and this factorization is unique in the sense that if f is also $m'e' : X \stackrel{e'}{\to} E' \stackrel{m'}{\to} Y$ with m' mono and e' regular epi, there is an isomorphism $\sigma : E \to E'$ such that $\sigma e = e'$ and $m'\sigma = m$.*

Proof. Given $f : X \to Y$ we let $X \xrightarrow{e} E$ be the coequalizer of the kernel pair $Z \overset{p_0}{\underset{p_1}{\rightrightarrows}} X$ of f. Since $fp_0 = fp_1$ there is a unique $m : E \to Y$ such that $f = me$. By construction e is regular epi; we must show that m is mono, and the uniqueness of the factorization.

Suppose $mg = mh$ for $g, h : W \to E$; we prove that $g = h$. Let

$$\begin{tikzcd} V \arrow[r, "a"] \arrow[d, "\langle q_0,q_1\rangle"'] & W \arrow[d, "\langle g,h\rangle"] \\ X \times X \arrow[r, "e\times e"'] & E \times E \end{tikzcd}$$

be a pullback square. Then

$$fq_0 = meq_0 = mga = mha = meq_1 = fq_1$$

so there is a unique arrow $V \xrightarrow{b} Z$ such that $\langle q_0, q_1\rangle = \langle p_0, p_1\rangle b : V \to X \times X$ (because of the kernel pair property). It follows that

$$ga = eq_0 = ep_0b = ep_1b = eq_1 = ha$$

I claim that a is epi, so it follows that $g = h$. It is here that we use the requirement that regular epis are stable under pullback. Now $e \times e : X \times X \to E \times E$ is the composite

$$X \times X \xrightarrow{e\times \mathrm{id}_X} E \times X \xrightarrow{\mathrm{id}_E\times e} E \times E$$

and both maps are regular epis since both squares

$$\begin{tikzcd} X \times X \arrow[r, "e\times\mathrm{id}_X"] \arrow[d, "\pi_0"'] & E \times X \arrow[d, "\pi_0"] \\ X \arrow[r, "e"'] & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \times X \arrow[r, "\mathrm{id}_E\times e"] \arrow[d, "\pi_1"'] & E \times E \arrow[d, "\pi_1"] \\ X \arrow[r, "e"'] & E \end{tikzcd}$$

are pullbacks. The map a, being the pullback of a composite of regular epis, is then itself the composite of regular epis (check this!), so in particular epi.

This proves that m is mono, and we have our factorization.

As to uniqueness, suppose we had another factorization $f = m'e'$ with m' mono and e' regular epi. Then $m'e'p_0 = fp_0 = fp_1 = m'e'p_1$ so since m' mono, $e'p_0 = e'p_1$. Because e is the coequalizer of p_0 and p_1, there is a unique σ:

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[dr, "e'"'] & \cdot \arrow[d, "\sigma"] \\ & \cdot \end{tikzcd}$$

Then $m'\sigma e = m'e' = f = me$ so since e epi, $m = m'\sigma$.

Now $e' : X \to E'$ is a coequalizer; say $U \overset{k}{\underset{l}{\rightrightarrows}} X \xrightarrow{e'} E'$ is a coequalizer diagram. Then it follows that $ek = el$ (since $mek = m'e'k = m'e'l = mel$

and m mono) so there is a unique τ: $\begin{tikzcd} {} \arrow[r, "e"] \arrow[dr, "e'"'] & {} \\ & {} \arrow[u, "\tau"'] \end{tikzcd}$ Then $m\tau\sigma e = m\tau e' = me$; since m mono and e epi, $\tau\sigma = \mathrm{id}_E$. Similarly, $\sigma\tau = \mathrm{id}_{E'}$. So σ is the required isomorphism. ∎

Exercise 66 Check this detail: in a regular category $\mathcal{C}$, if

$$\begin{tikzcd} {} \arrow[r, "a"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "b"'] & {} \end{tikzcd}$$

is a pullback diagram and $b = c_1c_2$ with c_1 and c_2 regular epis, then $a = a_1a_2$ for some regular epis a_1, a_2.

Proposition 4.3 *Let $\mathcal{C}$ be a regular category.*

a) *Suppose that*

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

is a pullback diagram in $\mathcal{C}$ with e regular epi. Then if g is mono, so is f.

b) *The composition of two regular epis in $\mathcal{C}$ is again regular epi in $\mathcal{C}$.*

Proof. For a), suppose $E \overset{p_0}{\underset{p_1}{\rightrightarrows}} B$ is a parallel pair for which $fp_0 = fp_1$. Let

$$\begin{tikzcd} F \arrow[r, "h'"] \arrow[d, "g'"'] & E \arrow[d, "fp_0"] \\ C \arrow[r, "e"'] & D \end{tikzcd}$$

be a pullback. Then by the pullback property of the original diagram there are arrows $q_0, q_1 : F \to A$ such that $gq_0 = g', hq_0 = p_0h'$ and $gq_1 = g', hq_1 = p_1h'$:

$$\begin{tikzcd} F \arrow[rr, "h'"] \arrow[dr, shift left, "q_1"] \arrow[dr, shift right, "q_0"'] \arrow[ddr, "g'"'] & & E \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] \\ & A \arrow[r, "h"] \arrow[d, "g"'] & B \arrow[d, "f"] \\ & C \arrow[r, "e"'] & D \end{tikzcd}$$

From $gq_0 = g' = gq_1$ and the assumption that g is mono, we get $q_0 = q_1$. Therefore $p_0h' = hq_0 = hq_1 = p_1h'$. Since h', being a pullback of the regular epi e, is regular epi (hence epi), we find $p_0 = p_1$. We conclude that f is mono.

For b), suppose in $A \overset{e_1}{\to} B \overset{e_2}{\to} C$ the arrows e_1, e_2 are both regular epi. In order to show that the composite e_2e_1 is regular epi, we factor this composite as me with m mono and e regular epi:

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[dr, "e"'] & B \arrow[r, "e_2"] & C \\ & D \arrow[ur, "m"'] & \end{tikzcd}$$

If $E \overset{p_0}{\underset{p_1}{\rightrightarrows}} A$ is the kernel pair of e_1 then $mep_0 = e_2e_1p_0 = e_2e_1p_1 = mep_1$ so since m in mono, $ep_0 = ep_1$. Therefore, since e_1 is the coequalizer of p_0, p_1 we have a unique map $n : B \to D$ satisfying $ne_1 = e$. Then we also have: $mne_1 = me = e_2e_1$, so since e_1 is epi, $mn = e_2$ and the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "e_1"] \arrow[dr, "e"'] & B \arrow[r, "e_2"] \arrow[d, "n"] & C \\ & D \arrow[ur, "m"'] & \end{tikzcd}$$

Repeating the argument for the kernel pair q_0, q_1 of e_2, we get that $nq_0 = nq_1$; so since e_2 is the coequalizer of its kernel pair, we get a unique arrow $k : C \to D$ such that $ke_2 = n$.

Then $mke_2 = mn = e_2$ so since e_2 is epi, $mk = \mathrm{id}_C$; and $kme = ke_2e_1 = ne_1 = e$, so since e is epi, $km = \mathrm{id}_D$. We find that k is a two-sided inverse for m, which is therefore an isomorphism. We conclude that e_2e_1 is regular epi. ∎

Subobjects. In any category $\mathcal{C}$ we define a *subobject* of an object X to be an equivalence class of monomorphisms $Y \overset{m}{\to} X$, where $Y \overset{m}{\to} X$ is equivalent to $Y' \overset{m'}{\to} X$ if there is an isomorphism $\sigma : Y \to Y'$ with $m'\sigma = m$ (then $m\sigma^{-1} = m'$ follows). We say that $Y \overset{m}{\to} X$ represents a *smaller* subobject than $Y' \overset{m'}{\to} X$ if there is $\sigma : Y \to Y'$ with $m'\sigma = m$ (σ not necessarily iso; but check that σ is always mono).

The notion of subobject is the categorical version of the notion of subset in set theory. In Set, two injective functions represent the same subobject iff their images are the same; one can therefore identify subobjects with subsets. Note however, that in Set we have a "canonical" *choice* of representative for each subobject: the inclusion of the subset to which the subobject corresponds. This choice is not always available in general categories.

We have a partial order $\mathrm{Sub}(X)$ of subobjects of X, ordered by the smaller-than relation.

Proposition 4.4 *In a category $\mathcal{C}$ with finite limits, each pair of elements of $\mathrm{Sub}(X)$ has a greatest lower bound. Moreover, $\mathrm{Sub}(X)$ has a largest element.*

Proof. If $Y \stackrel{m}{\to} X$ and $Y' \stackrel{m'}{\to} X$ represent two subobjects of X and
$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "m"] \\ Y' \arrow[r, "m'"'] & X \end{tikzcd}$$
is a pullback, then $Z \to X$ is mono, and represents the greatest lower bound (check!).

Of course, the identity $X \stackrel{\mathrm{id}_X}{\to} X$ represents the top element of $\mathrm{Sub}(X)$. ∎

Because the factorization of $X \stackrel{f}{\to} Y$ as $X \stackrel{e}{\to} E \stackrel{m}{\to} Y$ with e regular epi and m mono, in a regular category $\mathcal{C}$, is only defined up to isomorphism, it defines rather a subobject of Y than a mono into Y; this defined subobject is called the *image* of f and denoted $\mathrm{Im}(f)$ (compare with the situation in Set).

Exercise 67 $\mathrm{Im}(f)$ is the smallest subobject of Y through which f factors: for a subobject represented by $n : A \to Y$ we have that there is $X \stackrel{a}{\to} A$ with $f = na$, if and only if $\mathrm{Im}(f)$ is smaller than the subobject represented by n.

Since monos and isos are stable under pullback, in any category $\mathcal{C}$ with pullbacks, any arrow $f : X \to Y$ determines an order preserving map $f^* : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ *by pullback along* f: if $E \stackrel{m}{\to} Y$ represents the subobject M of Y and
$$\begin{tikzcd} F \arrow[r] \arrow[d, "n"'] & E \arrow[d, "m"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
is a pullback, $F \stackrel{n}{\to} X$ represents $f^*(M)$.

Exercise 68 Check that f^* is well defined and order preserving.

Proposition 4.5 *In a regular category, each f^* preserves greatest lower bounds and images, that is: for $f : X \to Y$,*

i) for subobjects M, N of Y, $f^*(M \wedge N) = f^*(M) \wedge f^*(N)$;

ii) if
$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, "g'"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"'] & \cdot \end{tikzcd}$$
is a pullback, then $f^*(\mathrm{Im}(g)) = \mathrm{Im}(g')$.

Exercise 69 Prove proposition 4.5.

Exercise 70 Suppose $f : X \to Y$ is an arrow in a regular category. For a subobject M of X, represented by a mono $E \stackrel{m}{\to} X$, write $\exists_f(M)$ for the subobject $\mathrm{Im}(fm)$ of Y.

a) Show that $\exists_f(M)$ is well-defined, that is: depends only on M, not on the representative m.

b) Show that if $M \in \mathrm{Sub}(X)$ and $N \in \mathrm{Sub}(Y)$, then $\exists_f(M) \le N$ if and only if $M \le f^*(N)$.

4.2 The logic of regular categories

The fragment of first order logic we are going to interpret in regular categories is the so-called regular fragment.

The logical symbols are $=$ (equality), $\wedge$ (conjunction) and $\exists$ (existential quantification). A *language* consists of a set of *sorts* $S, T, \ldots$; a denumerable collection of *variables* $x_1^S, x_2^S, \ldots$ of sort S, for each sort; a collection of *function symbols* $(f : S_1, \ldots, S_n \to S)$ and *relation symbols* $(R \subseteq S_1, \ldots, S_m)$. The case $n = 0$ is not excluded (one thinks of *constants* of a sort in case of 0-placed function symbols, and of *atomic propositions* in the case of 0-placed relation symbols), but not separately treated. We now define, inductively, terms of sort S and formulas.

Definition 4.6 *Terms of sort S are defined by:*

i) x^S *is a term of sort S if x^S is a variable of sort S;*

ii) *if $t_1, \ldots, t_n$ are terms of sorts $S_1, \ldots, S_n$ respectively, and*

$$(f : S_1, \ldots S_n \to S)$$

is a function symbol of the language, then $f(t_1, \ldots, t_n)$ is a term of sort S.

Formulas are defined by:

i) $\top$ *is a formula (the formula "true");*

ii) *if t and s are terms of the same sort, then $t = s$ is a formula;*

iii) *if $(R \subseteq S_1, \ldots, S_m)$ is a relation symbol and $t_1, \ldots, t_m$ are terms of sorts $S_1, \ldots, S_m$ respectively, then $R(t_1, \ldots, t_m)$ is a formula;*

iv) *if φ and ψ are formulas then $(\varphi \wedge \psi)$ is a formula;*

v) *if φ is a formula and x a variable of some sort, then $\exists x \varphi$ is a formula.*

An *interpretation* of such a language in a regular category $\mathcal{C}$ is given by choosing for each sort S an object $[\![S]\!]$ of $\mathcal{C}$, for each function symbol $(f : S_1, \ldots, S_n \to S)$ of the language, an arrow $[\![f]\!] : [\![S_1]\!] \times \cdots \times [\![S_n]\!] \to [\![S]\!]$ in $\mathcal{C}$, and for each relation symbol $(R \subseteq S_1, \ldots, S_m)$ a subobject $[\![R]\!]$ of $[\![S_1]\!] \times \cdots \times [\![S_m]\!]$.

Given this, we define interpretations $[\![t]\!]$ for terms t and $[\![\varphi]\!]$ for formulas φ, as follows.

Write $FV(t)$ for the set of variables which occur in t, and $FV(\varphi)$ for the set of *free* variables in φ.

We put $[\![FV(t)]\!] = [\![S_1]\!] \times \cdots \times [\![S_n]\!]$ if $FV(t) = \{x_1^{S_1}, \ldots, x_n^{S_n}\}$; the same for $[\![FV(\varphi)]\!]$. Note: in the products $[\![FV(t)]\!]$ and $[\![FV(\varphi)]\!]$ we take a copy of $[\![S]\!]$ for every variable of sort S! Let me further emphasize that the empty product is 1, so if $FV(t)$ (or $FV(\varphi)$) is $\emptyset$, $[\![FV(t)]\!]$ (or $[\![FV(\varphi)]\!]$) is the terminal object of the category.

Definition 4.7 *The interpretation of a term t of sort S is a morphism $[\![\, t \,]\!] : [\![\, FV(t) \,]\!] \to [\![\, S \,]\!]$ and is defined by the clauses:*

i) $[\![\, x^S \,]\!]$ *is the identity on* $[\![\, S \,]\!]$, *if x^S is a variable of sort S;*

ii) *Given* $[\![\, t_i \,]\!] : [\![\, FV(t_i) \,]\!] \to [\![\, S_i \,]\!]$ *for $i = 1, \dots, n$ and a function symbol $(f : S_1, \dots, S_n \to S)$ of the language,* $[\![\, f(t_1, \dots, t_n) \,]\!]$ *is the map*

$$[\![\, FV(f(t_1, \dots, t_n)) \,]\!] \xrightarrow{(\tilde{t}_i | i=1,\dots,n)} \prod_{i=1}^n [\![\, S_i \,]\!] \xrightarrow{[\![\, f \,]\!]} [\![\, S \,]\!]$$

where $\tilde{t}_i$ is the composite

$$[\![\, FV(f(t_1, \dots, t_n)) \,]\!] \xrightarrow{\pi_i} [\![\, FV(t_i) \,]\!] \xrightarrow{[\![\, t_i \,]\!]} [\![\, S_i \,]\!]$$

in which π_i is the appropriate projection, corresponding to the inclusion $FV(t_i) \subseteq FV(f(t_1, \dots, t_n))$.

Finally, we interpret formulas φ as *subobjects* $[\![\, \varphi \,]\!]$ of $[\![\, FV(\varphi) \,]\!]$. You should try to keep in mind the intuition that $[\![\, \varphi(x_1, \dots, x_n) \,]\!]$ is the "subset"

$$\{(a_1, \dots, a_n) \in A_1 \times \cdots \times A_n \,|\, \varphi[a_1, \dots, a_n]\}$$

Definition 4.8 *The interpretation* $[\![\, \varphi \,]\!]$ *as subobject of* $[\![\, FV(\varphi) \,]\!]$ *is defined as follows:*

i) $[\![\, \top \,]\!]$ *is the maximal subobject of* $[\![\, FV(\top) \,]\!] = 1$*;*

ii) $[\![\, t = s \,]\!] \to [\![\, FV(t = s) \,]\!]$ *is the equalizer of*

$$[\![\, FV(t = s) \,]\!] \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \begin{array}{c} [\![\, FV(t) \,]\!] \\ [\![\, FV(s) \,]\!] \end{array} \begin{array}{c} \xrightarrow{[\![\, t \,]\!]} \\ \xrightarrow[{[\![\, s \,]\!]}]{} \end{array} [\![\, T \,]\!]$$

if t and s are of sort T; again, the left hand side maps are projections, corresponding to the inclusions of $FV(t)$ and $FV(s)$ into $FV(t = s)$;

iii) *For $(R \subseteq S_1, \dots, S_m)$ a relation symbol and terms $t_1, \dots, t_m$ of sorts $S_1, \dots, S_m$ respectively, let* $\bar{t} : [\![\, FV(R(t_1, \dots, t_m)) \,]\!] \to \prod_{i=1}^m [\![\, S_i \,]\!]$ *be the composite map*

$$[\![\, FV(R(t_1, \dots, t_m)) \,]\!] \to \prod_{i=1}^m [\![\, FV(t_i) \,]\!] \overset{\prod_{i=1}^m [\![\, t_i \,]\!]}{\to} \prod_{i=1}^m [\![\, S_i \,]\!]$$

Then $[\![\, R(t_1, \dots, t_m) \,]\!] \to [\![\, FV(R(t_1, \dots, t_m)) \,]\!]$ *is the subobject* $(\bar{t})^*([\![\, R \,]\!])$*, defined by pullback along $\bar{t}$.*

iv) *if* $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ *and* $[\![\psi]\!] \to [\![FV(\psi)]\!]$ *are given and*

$$\begin{tikzcd} {[\![FV(\varphi\wedge\psi)]\!]} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & {[\![FV(\varphi)]\!]} \\ & {[\![FV(\psi)]\!]} \end{tikzcd}$$

are again the suitable projections, then $[\![(\varphi\wedge\psi)]\!] \to [\![FV(\varphi\wedge\psi)]\!]$ *is the greatest lower bound in* $\mathrm{Sub}([\![FV(\varphi\wedge\psi)]\!])$ *of* $\pi_1^*([\![\varphi]\!])$ *and* $\pi_2^*([\![\psi]\!])$*;*

v) *if* $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ *is given and* $\pi : [\![FV(\varphi)]\!] \to [\![FV(\exists x\varphi)]\!]$ *the projection, let* $[\![FV'(\varphi)]\!]$ *be the product of the interpretations of the sorts of the variables in* $FV(\varphi)\cup\{x\}$ *(so* $[\![FV'(\varphi)]\!] = [\![FV(\varphi)]\!]$ *if* x *occurs freely in* φ*; and* $[\![FV'(\varphi)]\!] = [\![FV(\varphi)]\!] \times [\![S]\!]$ *if* $x = x^S$ *does not occur free in* φ*). Write* $\pi' : [\![FV'(\varphi)]\!] \to [\![FV(\varphi)]\!]$*.*

Now take $[\![\exists x\varphi]\!] \to [\![FV(\exists x\varphi)]\!]$ *to be the image of the composition:*

$$(\pi')^*([\![\varphi]\!]) \to [\![FV'(\varphi)]\!] \stackrel{\pi\pi'}{\to} [\![FV(\exists x\varphi)]\!]$$

We have now given an interpretation of formulas. Basically, a formula φ is *true* under this interpretation if $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ is the maximal subobject; but since we formulate the logic in terms of sequents we rather define when a sequent is true under the interpretation.

Definition 4.9 *A* labelled sequent *is an expression of the form* $\psi\vdash_\sigma\varphi$ *or* $\vdash_\sigma\varphi$ *where* ψ *and* φ *are the formulas of the sequent (but* ψ *may be absent), and* σ *is a finite set of variables which includes all the variables which occur free in a formula of the sequent.*

Let $[\![\sigma]\!] = [\![S_1]\!] \times\cdots\times [\![S_n]\!]$ *if* $\sigma = \{x_1^{S_1},\ldots,x_n^{S_n}\}$*; there are projections* $[\![\sigma]\!] \stackrel{\pi_\varphi}{\to} [\![FV(\varphi)]\!]$ *and (in case* ψ *is there)* $[\![\sigma]\!] \stackrel{\pi_\psi}{\to} [\![FV(\psi)]\!]$*; we say that the sequent* $\psi\vdash_\sigma\varphi$ *is true for the interpretation if* $(\pi_\psi)^*([\![\psi]\!]) \leq (\pi_\varphi)^*([\![\varphi]\!])$ *as subobjects of* $[\![\sigma]\!]$*, and* $\vdash_\sigma\varphi$ *is true if* $(\pi_\varphi)^*([\![\varphi]\!])$ *is the maximal subobject of* $[\![\sigma]\!]$*.*

We also say that φ *is true if* $\vdash_{\mathrm{FV}(\varphi)}\varphi$ *is true.*

Exercise 71 Show that the sequent $\vdash\exists x^S(x^S = x^S)$ is true if and only if the unique map $[\![S]\!] \to 1$ is a regular epimorphism. What about the sequent $\vdash_S\top$?

We now turn to the logic. Instead of giving deduction rules and axioms, I formulate a list of closure conditions which determine what sets of labelled sequents will be called a *theory.* I write $\vdash_x$ for $\vdash_{\{x\}}$ and $\vdash$ for $\vdash_\emptyset$.

Definition 4.10 *Given a language, a set* T *of labelled sequents of that language is called a* theory *iff the following conditions hold (the use of brackets around* ψ *caters in a, I hope, self-explanatory way for the case distiction as to whether* ψ *is or is not present):*

i) $\vdash \top$ *is in* T*;*
$\vdash_x x = x$ *is in* T *for every variable* x*;*
$x = y \vdash_{\{x,y\}} y = x$ *is in* T *for variables* x, y *of the same sort;*
$x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$ *is in* T *for variables* x, y, z *of the same sort;*
$R(x_1, \dots, x_m) \vdash_{\{x_1,\dots,x_m\}} R(x_1, \dots, x_m)$ *is in* T*;*

ii) *if* $(\psi) \vdash_\sigma \varphi$ *is in* T *then* $(\psi) \vdash_\tau \varphi$ *is in* T *whenever* $\sigma \subseteq \tau$*;*

iii) *if* $(\psi) \vdash_\sigma \varphi$ *is in* T *and* $FV(\chi) \subseteq \sigma$ *then* $(\psi\wedge)\chi \vdash_\sigma \varphi$ *and* $\chi(\wedge\psi) \vdash_\sigma \varphi$ *are in* T*;*

iv) *if* $(\psi) \vdash_\sigma \varphi$ *and* $(\psi) \vdash_\sigma \chi$ *are in* T *then* $(\psi) \vdash_\sigma \varphi \wedge \chi$ *and* $(\psi) \vdash_\sigma \chi \wedge \varphi$ *are in* T*;*

v) *if* $(\chi\wedge)\psi \vdash_\sigma \varphi$ *is in* T *and* x *is a variable not occurring in* φ *or* χ *then* $(\chi\wedge)\exists x\psi \vdash_{\sigma\setminus\{x\}} \varphi$ *is in* T*;*

vi) *if* x *occurs in* φ *and* $(\psi) \vdash_\sigma \varphi[t/x]$ *is in* T *then* $(\psi) \vdash_\sigma \exists x\varphi$ *is in* T*;*
if x *does not occur in* φ *and* $(\psi) \vdash_\sigma \varphi$ *and* $(\psi) \vdash_\sigma \exists x(x = x)$ *are in* T*, then* $(\psi) \vdash_\sigma \exists x\varphi$ *is in* T*;*

vii) *if* $(\psi) \vdash_\sigma \varphi$ *is in* T *then* $(\psi[t/x]) \vdash_{\sigma\setminus\{x\}\cup FV(t)} \varphi[t/x]$ *is in* T*;*

viii) *if* $(\psi) \vdash_\sigma \varphi[t/x]$ *and* $(\psi) \vdash_\sigma t = s$ *are in* T *then* $(\psi) \vdash_\sigma \varphi[s/x]$ *is in* T*;*

ix) *if* $(\psi) \vdash_\sigma \varphi$ *and* $\varphi \vdash_\sigma \chi$ *are in* T *then* $(\psi) \vdash_\sigma \chi$ *is in* T

Exercise 72 Show that the sequent $\varphi \vdash_{FV(\varphi)} \varphi$ is in every theory, for every formula φ of the language.

As said, the definition of a theory is a list of closure conditions: every item can be seen as a rule, and a theory is a set of sequents closed under every rule. Therefore, the intersection of any collection of theories is again a theory, and it makes sense to speak, given a set of sequents S, of the theory $Cn(S)$ *generated by* S:

$$Cn(S) = \bigcap\{T | T \text{ is a theory and } S \subseteq T\}$$

We have the following theorem:

Theorem 4.11 (Soundness theorem) *Suppose* $T = Cn(S)$ *and all sequents of* S *are true under the interpretation in the category* $\mathcal{C}$*. Then all sequents of* T *are true under that interpretation.*

Before embarking on the proof, first a lemma:

Lemma 4.12 *Suppose* t *is substitutable for* x *in* φ*. There is an obvious map*

$$[t] : [\![FV(\varphi) \setminus \{x\} \cup FV(t)]\!] = [\![FV(\varphi[t/x])]\!] \to [\![FV(\varphi)]\!]$$

induced by $[\![\, t \,]\!]$*; the components of* $[t]$ *are projections except for the factor of* $[\![\, \varphi \,]\!]$ *corresponding to* x*, where it is*

$$[\![\, FV(\varphi[t/x]) \,]\!] \to [\![\, FV(t) \,]\!] \stackrel{[\![t]\!]}{\to} [\![\, \{x\} \,]\!]$$

There is a pullback diagram:

$$\begin{tikzcd}
{[\![\, \varphi[t/x] \,]\!]} \arrow[r] \arrow[d] & {[\![\, FV(\varphi[t/x]) \,]\!]} \arrow[d, "{[t]}"] \\
{[\![\, \varphi \,]\!]} \arrow[r] & {[\![\, FV(\varphi) \,]\!]}
\end{tikzcd}$$

Exercise 73 Prove this lemma [not trivial. Use induction on φ and proposition 4.5].

Proof. (of theorem 4.11) The proof checks that for every rule in the list of definition 4.10, if the premiss is true then the conclusion is true; in other words, that the set of true sequents is a theory.
i) $\vdash \top$ is true by the definition $[\![\, \top \,]\!] = 1$;
$[\![\, x^S = x^S \,]\!]$ is the equalizer of two maps which are both the identity on $[\![\, S \,]\!]$, so isomorphic to $[\![\, S \,]\!]$. For $x = y \wedge y = z \vdash_{\{x,y,z\}} x = z$, just observe that $E_{12} \wedge E_{23} \leq E_{13}$ if E_{ij} is the equalizer of the two projections $\pi_i, \pi_j : [\![\, S \,]\!] \times [\![\, S \,]\!] \times [\![\, S \,]\!] \to [\![\, S \,]\!]$.
ii) This is because if $\sigma \subseteq \tau$ and $\rho : [\![\, \tau \,]\!] \to [\![\, \sigma \,]\!]$ is the projection, ρ^* is monotone.
iii)-iv) By the interpretation of $\wedge$ as the greatest lower bound of two subobjects, and proposition 4.5.
v) Let

$$\begin{tikzcd}
{[\![\, \sigma \,]\!]} \arrow[r, "\pi"] \arrow[d, "\mu"'] & {[\![\, \sigma \setminus \{x\} \,]\!]} \arrow[r, "\rho"] \arrow[d, "\nu"] & {[\![\, FV(\varphi) \,]\!]} \\
{[\![\, FV(\psi) \,]\!]} & {[\![\, FV(\exists x\psi) \,]\!]} &
\end{tikzcd}$$

the projections. Since by assumption $\mu^*([\![\, \psi \,]\!]) \leq (\rho\pi)^*([\![\, \varphi \,]\!])$ there is a commutative diagram

$$\begin{tikzcd}
{\mu^*([\![\, \psi \,]\!])} \arrow[r] \arrow[d] & {[\![\, \sigma \,]\!]} \arrow[d, "\pi"] \\
{\rho^*([\![\, \varphi \,]\!])} \arrow[r] & {[\![\, \sigma \setminus \{x\} \,]\!]}
\end{tikzcd}$$

By proposition 4.5, $\nu^*([\![\, \exists x\psi \,]\!])$ is the image of the map $\mu^*([\![\, \psi \,]\!]) \to [\![\, \sigma \setminus \{x\} \,]\!]$, so $\nu^*([\![\, \psi \,]\!]) \leq \rho^*([\![\, \varphi \,]\!])$ in $\mathrm{Sub}([\![\, \sigma \setminus \{x\} \,]\!])$.
vi) Suppose x occurs free in φ. Consider the commutative diagram

$$\begin{tikzcd}
 & {[\![\, \sigma \,]\!]} \arrow[dl, "\pi"'] \arrow[d, "\pi'"] \arrow[drrr, "\pi''"] & & & \\
{[\![\, FV(\psi) \,]\!]} & {[\![\, FV(\varphi[t/x]) \,]\!]} \arrow[r, "{[t]}"] & {[\![\, FV(\varphi) \,]\!]} \arrow[rr, "\rho"] & & {[\![\, FV(\varphi) \setminus \{x\} \,]\!]}
\end{tikzcd}$$

with $[t]$ as in lemma 4.12 and the other maps projections. Now $[\![\varphi]\!] \leq \rho^*([\![\exists x\varphi]\!])$ because $[\![\varphi]\!] \to [\![FV(\varphi)]\!] \xrightarrow{\rho} [\![FV(\varphi) \setminus \{x\}]\!]$ factors through $[\![\exists x\varphi]\!]$ by definition; so if $\pi^*([\![\psi]\!]) \leq \pi'^*([\![\varphi[t/x]]\!])$ then with lemma 4.12,

$$\pi^*([\![\psi]\!]) \leq \pi'^*([\![\varphi[t/x]]\!]) = \pi'^*[t]^*([\![\varphi]\!]) \leq \pi'^*[t]^*\rho^*([\![\exists x\varphi]\!]) = \pi''^*([\![\exists x\varphi]\!])$$

in $\mathrm{Sub}([\![\sigma]\!])$ and we are done.

The case of x not occurring in φ is left to you.

vii) Direct application of lemma 4.12

viii-ix) Left to you. ∎

Exercise 74 Fill in the "left to you" gaps in the proof.

4.3 The language $\mathcal{L}(\mathcal{C})$ and theory $T(\mathcal{C})$ associated to a regular category $\mathcal{C}$

Given a regular category $\mathcal{C}$ (which, to be precise, must be assumed to be small), we associate to $\mathcal{C}$ the language which has a sort C for every object of $\mathcal{C}$, and a function symbol $(f : C \to D)$ for every arrow $f : C \to D$ of $\mathcal{C}$.

This language is called $\mathcal{L}(\mathcal{C})$ and it has trivially an interpretation in $\mathcal{C}$.

The theory $T(\mathcal{C})$ is the set of sequents of $\mathcal{L}(\mathcal{C})$ which are true for this interpretation.

One of the points of categorical logic is now, that categorical statements about objects and arrows in $\mathcal{C}$ can be reformulated as statements about the truth of certain sequents in $\mathcal{L}(\mathcal{C})$. You should read the relevant sequents as expressing that we can "do as if the category were Set".

Examples

a) C is a terminal object of $\mathcal{C}$ if and only if the sequents $\vdash_{x,y} x = y$ and $\vdash \exists x(x = x)$ are valid, where x, y variables of sort C;

b) the arrow $f : A \to B$ is mono in $\mathcal{C}$ if and only if the sequent $f(x) = f(y) \vdash_{x,y} x = y$ is true;

c) The square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "d"'] & D \end{tikzcd}$$

is a pullback square in $\mathcal{C}$ if and only if the sequents

$$h(x^B) = d(y^C) \vdash_{x,y} \exists z^A(f(z) = x \wedge g(z) = y)$$

and

$$f(z^A) = f(z'^A) \wedge g(z^A) = g(z'^A) \vdash_{z,z'} z = z'$$

are true;

d) the fact that $f : A \to B$ is epi can not similarly be expressed! But: f is regular epi if and only if

$$\vdash_{x^B} \exists y^A(f(y) = x)$$

is true;

e) $A \xrightarrow{f} B \overset{g}{\underset{h}{\rightrightarrows}} C$ is an equalizer diagram iff f is mono (see b) and the sequent

$$g(x^B) = h(x^B) \vdash_{x^B} \exists y^A(f(y) = x)$$

is true.

Exercise 75 Check (a number of) these statements. Give the sequent(s) corresponding to the statement that

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C \end{tikzcd}$$

is a product diagram.

Exercise 76 Check that the formulas $\exists x\varphi$ and $\exists x(x = x \wedge \varphi)$ are *equivalent*, that is, every theory contains the sequents

$$\exists x\varphi \vdash_\sigma \exists x(x = x \wedge \varphi)$$

and

$$\exists x(x = x \wedge \varphi) \vdash_\sigma \exists x\varphi$$

for any σ containing the free variables of $\exists x\varphi$.

Exercise 77 Suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "d"'] & D \end{tikzcd}$$

is a pullback diagram in a regular category $\mathcal{C}$, with d regular epi. Prove the result of Proposition 4.3 a) (that is, if g is mono, then so is h), by using regular logic. Can you say anything about the relation between the two proofs?

Exercise 78 Can you express: $A \xrightarrow{f} B$ is regular mono? [Hint: don't waste too much time in trying!]

4.4 The category $\mathcal{C}(T)$ associated to a theory T: Completeness Theorem

The counterpart of Theorem 4.11 (the Soundness Theorem) is of course a completeness theorem: suppose that the sequent $(\psi) \vdash_\sigma \varphi$ is true in every interpretation which makes all sequents from T true. We want to conclude that $(\psi) \vdash_\sigma \varphi$ is in T.

To do this, we build a category $\mathcal{C}(T)$, a so-called *syntactic category*, which will be regular, and which allows an interpretation $[\![\cdot]\!]$ such that exactly the sequents in T will be true for $[\![\cdot]\!]$.

The construction of $\mathcal{C}(T)$ is as follows. *Objects* are formulas of the language $\mathcal{L}$, up to renaming of free variables; so if φ is a formula in distinct free variables $x_1, \ldots, x_n$ and $v_1, \ldots, v_n$ are distinct variables of matching sorts, then φ and $\varphi(v_1/x_1, \ldots, v_n/x_n)$ are the same object. Because of this stipulation, if I define morphisms between $\varphi(\vec{x})$ and $\psi(\vec{y})$, I can always assume that the collections of variables $\vec{x}$ and $\vec{y}$ are disjoint (even when treating a morphism from φ to itself, I take φ and a renaming $\varphi(\vec{x'})$). Given $\varphi(\vec{x})$ and $\psi(\vec{y})$ (where $\vec{x} = (x_1, \ldots, x_n)$, $\vec{y} = (y_1, \ldots, y_m)$), a *functional relation* from φ to ψ is a formula $\chi(\vec{x}, \vec{y})$ such that the sequents

$$\begin{gathered}
\chi(\vec{x}, \vec{y}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m\}} \varphi(\vec{x}) \wedge \psi(\vec{y}) \\
\chi(\vec{x}, \vec{y}) \wedge \chi(\vec{x}, \vec{y'}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m,y'_1,\ldots,y'_m\}} y_1 = y'_1 \wedge \cdots \wedge y_m = y'_m \\
\varphi(\vec{x}) \vdash_{\{x_1,\ldots,x_n\}} \exists y_1 \cdots \exists y_m \, \chi(\vec{x}, \vec{y})
\end{gathered}$$

are all in T. If $FV(\psi) = \emptyset$, the second requirement is taken to be vacuous, i.e. trivially fulfilled.

A *morphism* from φ to ψ is an equivalence class $[\chi]$ of functional relations from φ to ψ, where χ_1 and χ_2 are equivalent iff the sequents

$$\begin{gathered}
\chi_1(\vec{x}, \vec{y}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m\}} \chi_2(\vec{x}, \vec{y}) \\
\chi_2(\vec{x}, \vec{y}) \vdash_{\{x_1,\ldots,x_n,y_1,\ldots,y_m\}} \chi_1(\vec{x}, \vec{y})
\end{gathered}$$

are in T (in fact, given that χ_1 and χ_2 are functional relations, one of these sequents is in T iff the other is).

Composition of morphisms is defined as follows: if $\chi_1(\vec{x}, \vec{y})$ represents a morphism $\varphi \to \psi$ and $\chi_2(\vec{y}, \vec{z})$ a morphism $\psi \to \omega$, the composition $[\chi_2] \circ [\chi_1] : \varphi \to \omega$ is represented by the functional relation

$$\chi_{21} \equiv \exists y_1 \cdots \exists y_m \, (\chi_1(\vec{x}, \vec{y}) \wedge \chi_2(\vec{y}, \vec{z}))$$

Exercise 79 Show:

a) χ_{21} is a functional relation from φ to ω;

b) the class of χ_{21} does not depend on the choice of representatives χ_1 and χ_2;

c) composition is associative.

The *identity* arrow from $\varphi(\vec{x})$ to itself (i.e. to $\varphi(\vec{x'})$, given our renaming convention), is represented by the formula

$$\varphi(\vec{x}) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$$

Exercise 80 Show that this definition is correct: that this formula is a functional relation, and defines an identity arrow.

We have defined the category $\mathcal{C}(T)$.

Theorem 4.13 *$\mathcal{C}(T)$ is a regular category.*

Proof. The formula $\top$ is a terminal object in $\mathcal{C}(T)$: for every formula φ, φ itself represents the unique morphism $\varphi \to \top$. Given formulas $\varphi(\vec{x})$ and $\psi(\vec{y})$, the formula $\varphi \wedge \psi$ is a product, with projections $\varphi \wedge \psi \to \varphi(\vec{x'})$ (renaming!) represented by the formula

$$\varphi(\vec{x}) \wedge \psi(\vec{y}) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$$

and $\varphi \wedge \psi$ to $\psi(\vec{y'})$ similarly defined.

If $\chi_1(\vec{x}, \vec{y})$ and $\chi_2(\vec{x}, \vec{y})$ represent morphisms $\varphi(\vec{x}) \to \psi(\vec{y})$ let $\omega(\vec{x})$ be the formula

$$\exists y_1 \cdots \exists y_m(\chi_1(\vec{x}, \vec{y}) \wedge \chi_2(\vec{x}, \vec{y}))$$

Then $\omega(\vec{x}) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$ represents a morphism $\omega \to \varphi(\vec{x'})$ which is the equalizer of $[\chi_1]$ and $[\chi_2]$.

This takes care of finite limits.

Now if $[\chi_1(\vec{x}, \vec{y})] : \varphi(\vec{x}) \to \psi_1(\vec{y})$ and $[\chi_2(\vec{x}, \vec{y})] : \varphi(\vec{x}) \to \psi_2(\vec{y})$, then you can check that $[\chi_2]$ coequalizes the kernel pair of $[\chi_1]$ if and only if the sequent

$$\exists \vec{y}\,(\chi_1(\vec{x}, \vec{y}) \wedge \chi_1(\vec{x'}, \vec{y})) \vdash_{\{\vec{x}, \vec{x'}\}} \exists \vec{v}(\chi_2(\vec{x}, \vec{v}) \wedge \chi_2(\vec{x'}, \vec{v}))$$

is in T. This is the case if and only if $[\chi_2]$ factors through the obvious map from φ to $\exists \vec{x} \chi_1(\vec{x}, \vec{y})$ which is therefore the image of $[\chi_1]$, i.e. the coequalizer of its kernel pair.

We see at once that $[\chi] : \varphi \to \psi$ is regular epi iff the sequent

$$\psi(\vec{y}) \vdash_{\{\vec{y}\}} \exists \vec{x}\, \chi(\vec{x}, \vec{y})$$

is in T; using the description of finite limits one easily checks that these are stable under pullback ■

We define $[\![\cdot]\!]$, the standard interpretation of $\mathcal{L}$ in $\mathcal{C}(T)$, as follows:

- the interpretation $[\![S]\!]$ of a sort S is the formula $x = x$, where x is a variable of sort S;
- if $(f : S_1, \ldots, S_n \to S)$ is a function symbol of $\mathcal{L}$, $[\![f]\!]$ is the functional relation

$$f(x_1, \ldots, x_n) = x$$

- if $(R \subseteq S_1, \ldots, S_n)$ is a relation symbol of $\mathcal{L}$, the subobject $[\![R]\!]$ of $[\![S_1]\!] \times \cdots \times [\![S_n]\!]$ is represented by the mono

$$R(x_1, \ldots, x_n) \wedge x_1 = x_1' \wedge \cdots \wedge x_n = x_n'$$

I now state the important facts about $\mathcal{C}(T)$ and $[\![\cdot]\!]$ as nontrivial exercises. Here we say that, given the theory T, formulas φ and ψ (in the same free variables $x_1, \ldots, x_n$ are *equivalent in* T if the sequents $\varphi \vdash_{\{x_1,\ldots,x_n\}} \psi$ and $\psi \vdash_{\{x_1,\ldots,x_n\}} \varphi$ are both in T.

Exercise 81 If t is a term of sort S in $\mathcal{L}$, in variables $x_1^{S_1}, \ldots, x_n^{S_n}$, the functional relation from $[\![S_1]\!] \times \cdots \times [\![S_n]\!] \equiv x_1 = x_1 \wedge \cdots \wedge x_n = x_n$, to $[\![S]\!] \equiv x = x$, representing $[\![t]\!]$, is equivalent in T to the formula

$$t(x_1, \ldots, x_n) = x$$

Exercise 82 If φ is a formula of $\mathcal{L}$ in free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, the subobject $[\![\varphi]\!] \to [\![S_1]\!] \times \cdots \times [\![S_n]\!]$ is represented by a functional relation χ from a formula ψ to $x_1 = x_1 \wedge \cdots \wedge x_n = x_n$, such that:

a) ψ is a formula in variables $x_1', \ldots, x_n'$ and $\psi(x_1', \ldots, x_n')$ is equivalent in T to $\varphi(x_1', \ldots, x_n')$;

b) $\chi(x_1', \ldots, x_n', x_1, \ldots, x_n)$ is equivalent in T to

$$\varphi(x_1', \ldots, x_n') \wedge x_1' = x_1 \wedge \cdots \wedge x_n' = x_n$$

Exercise 83 The sequent $(\psi) \vdash_\sigma \varphi$ is true in the interpretation $[\![\cdot]\!]$ if and only if $(\psi) \vdash_\sigma \varphi$ is in T.

Exercise 84 Let $\mathcal{E}$ be a regular category and T a theory. Call a functor between regular categories *regular* if it preserves finite limits and regular epis.

Then every regular functor: $\mathcal{C}(T) \to \mathcal{E}$ gives rise to an interpretation of the language of T in $\mathcal{E}$, which makes all sequents in T true.

Conversely, given such an interpretation of T in $\mathcal{E}$, there is, up to natural isomorphism, a unique regular functor: $\mathcal{C}(T) \to \mathcal{E}$ with the property that it maps the standard interpretation of T in $\mathcal{C}(T)$ to the given one in $\mathcal{E}$.

Exercise 85 This exercise constructs the "free regular category on a given category $\mathcal{C}$". Given $\mathcal{C}$, which is not assumed to be regular (or to have finite limits), let $\mathcal{L}$ be the language of $\mathcal{C}$ as before: it has a sort C for every object C of $\mathcal{C}$, and a function symbol $(f : C \to D)$ for every arrow of $\mathcal{C}$. Let T be the theory generated by the following set S of sequents: for every identity arrow $i : C \to C$ in $\mathcal{C}$, the sequent $\vdash_x x = i(x)$ is in S; and for every composition $f = gh$ of arrows in $\mathcal{C}$, the sequent $\vdash_x f(x) = g(h(x))$ is in S.

a) Show that interpretations of $\mathcal{L}$ in a regular category $\mathcal{E}$ which make all sequents of S true, correspond bijectively to functors from $\mathcal{C}$ to $\mathcal{E}$.

b) Show that there is a functor $\eta : \mathcal{C} \to \mathcal{C}(T)$ such that for every functor $F : \mathcal{C} \to \mathcal{E}$ with $\mathcal{E}$ regular, there is, up to isomorphism, a unique regular functor $\tilde{F} : \mathcal{C}(T) \to \mathcal{E}$ such that $\tilde{F}\eta = F$. $\mathcal{C}(T)$ is the *free regular category on $\mathcal{C}$*.

c) Show that if, in this situation, $\mathcal{C}$ has finite limits, η does not preserve them! How would one construct the "free regular category on $\mathcal{C}$, preserving the limits which exist in $\mathcal{C}$"?

4.5 Example of a regular category

In this section, I treat an example of a type of regular categories which are important in categorical logic. They are categories of Ω-valued sets for some frame Ω. Let's define some things.

Definition 4.14 *A* frame Ω *is a partially ordered set which has suprema (least upper bounds) $\bigvee B$ of all subsets B, and infima (meets) $\bigwedge A$ for* finite *subsets A (so, there is a top element $\top$ and every pair of elements x, y has a meet $x \wedge y$), and moreover, $\wedge$ distributes over $\bigvee$, that is,*

$$x \wedge \bigvee B = \bigvee \{x \wedge b | b \in B\}$$

for $x \in \Omega$, $B \subseteq \Omega$.

Given a frame Ω we define the category $\mathcal{C}_\Omega$ as follows:
Objects are functions $X \stackrel{E_X}{\to} \Omega$, X a set;
Maps from (X, E_X) to (Y, E_Y) are functions $X \stackrel{f}{\to} Y$ satisfying the requirement that $E_X(x) \leq E_Y(f(x))$ for all $x \in X$.
It is easily seen that the identity function satisfies this requirement, and if two composable functions satisfy it, their composition does; so we have a category.

Proposition 4.15 *$\mathcal{C}_\Omega$ is a regular category.*

Proof. Let $\{*\}$ be any one-element set, together with the function which sends $*$ to the top element of Ω. Then $\{*\} \to \Omega$ is a terminal object of $\mathcal{C}_\Omega$.

Given (X, E_X) and (Y, E_Y), a product of the two is the object $(X \times Y, E_{X\times Y})$ where $E_{X\times Y}(x, y)$ is defined as $E_X(x) \wedge E_Y(y)$.

Given two arrows $f, g : (X, E_X) \to (Y, E_Y)$ their equalizer is $(X', E_{X'})$ where $X' \subseteq X$ is $\{x \in X | f(x) = g(x)\}$ and $E_{X'}$ is the restriction of E_X to X'.

This is easily checked, and $\mathcal{C}_\Omega$ is a finitely complete category.

An arrow $f : (X, E_X) \to (Y, E_Y)$ is regular epi if and only if the function f is surjective and for all $y \in Y$, $E_Y(y) = \bigvee\{E_X(x) \mid f(x) = y\}$.

For suppose f is such, and $g : (X, E_X) \to (Z, E_Z)$ coequalizes the kernel pair of f. Then $g(x) = g(x')$ whenever $f(x) = f(x')$, and so for all $y \in Y$, since $f(x) = y$ implies $E_X(x) \leq E_Z(g(x))$, we have

$$E_Y(y) = \bigvee\{E_X(x)|f(x) = y\} \leq E_Z(g(x))$$

so there is a unique map $h : (Y, E_Y) \to (Z, E_Z)$ such that $g = hf$; that is f is the coequalizer of its kernel pair.

The proof of the converse is left to you.

Finally we must show that regular epis are stable under pullback. This is an exercise. ∎

Exercise 86 Show that the pullback of
$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$
(I suppress reference to the E_X etc.) is (up to isomorphism) the set $\{(x, y) | f(x) = g(y)\}$, with $E(x, y) = E_X(x) \wedge E_Y(y)$; and then, use the distributivity of Ω to show that regular epis are stable under pullback.

Exercise 87 Fill in the other gap in the proof: if $f : (X, E_X) \to (Y, E_Y)$ is a regular epi, then f satisfies the condition given in the proof.

Exercise 88 Given $(X, E_X) \xrightarrow{f} (Y, E_Y)$, give the interpretation of the formula $\exists x(f(x) = y)$, as subobject of (Y, E_Y).

Exercise 89 Characterize those objects (X, E_X) for which the unique map to the terminal object is a regular epimorphism.

Exercise 90 Give a categorical proof of the statement: if f is the coequalizer of something, it is the coequalizer of its kernel pair.

Exercise 91 Characterize the regular monos in $\mathcal{C}_\Omega$.

Exercise 92 For every element u of Ω, let 1_u be the object of $\mathcal{C}_\Omega$ which is the function from a one-element set into Ω, with value u. Prove that every object of $\mathcal{C}_\Omega$ is a coproduct of objects of form 1_u.

Exercise 93 Let Ω be a frame. We consider the category $\mathcal{C}_\Omega$ and the presheaf category $\mathrm{Set}^{\Omega^{\mathrm{op}}}$.

We have the Yoneda embedding $y_{(-)} : \Omega \to \mathrm{Set}^{\Omega^{\mathrm{op}}}$ and we have a functor $H : \Omega \to \mathcal{C}_\Omega$, which sends $u \in \Omega$ to the object 1_u (in the notation of the previous exercise).

a) Show that there is an essentially unique functor $F : \mathcal{C}_\Omega \to \mathrm{Set}^{\Omega^{\mathrm{op}}}$ which preserves all small coproducts and moreover makes the diagram

$$\begin{tikzcd} \Omega \arrow[d, "H"'] \arrow[dr, "y"] & \\ \mathcal{C}_\Omega \arrow[r, "F"'] & \mathrm{Set}^{\Omega^{\mathrm{op}}} \end{tikzcd}$$

commute. Give a concrete description of $F(X, E_X)$ as a presheaf on Ω.

b) Suppose Ω has a nonempty subset B with the property that $\bigvee B \notin B$. Show that the functor F does not preserve regular epis.

5 Adjunctions

The following kind of problem occurs quite regularly: suppose we have a functor $\mathcal{C} \xrightarrow{G} \mathcal{D}$ and for a given object D of $\mathcal{D}$, we look for an object $G(C)$ which "best approximates" D, in the sense that there is a map $D \xrightarrow{\eta} G(C)$ with the property that any other map $D \xrightarrow{g} G(C')$ factors uniquely as $G(f)\eta$ for $f : C \to C'$ in $\mathcal{C}$.

We have seen, that if G is the inclusion of Abgp into Grp, the abelianization of a group is an example. Another example is the Čech-Stone compactification in topology: for a completely regular topological space X one constructs a compact Hausdorff space βX out of it, and a map $X \to \beta X$, such that any continuous map from X to a compact Hausdorff space factors uniquely through this map.

Of course, what we described here is a sort of "right-sided" approximation; the reader can define for himself what the notion for a left-sided approxiamtion would be.

The categorical description of this kind of phenomena goes via the concept of *adjunction*, which this chapter is about.

5.1 Adjoint functors

Let $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ be a pair of functors between categories $\mathcal{C}$ and $\mathcal{D}$.

We say that F is *left adjoint* to G, or G is *right adjoint* to F, notation: $F \dashv G$, if there is a natural bijection:

$$\mathcal{C}(FD, C) \xrightarrow{m_{D,C}} \mathcal{D}(D, GC)$$

for each pair of objects $C \in \mathcal{C}_0, D \in \mathcal{D}_0$. Two maps $f : FD \to C$ in $\mathcal{C}$ and $g : D \to GC$ in $\mathcal{D}$ which correspond to each other under this correspondence are called *transposes* of each other.

The naturality means that, given $f : D \to D'$, $g : C' \to C$ in $\mathcal{D}$ and $\mathcal{C}$ respectively, the diagram

$$\begin{tikzcd}
\mathcal{C}(FD, C) \arrow[r, "m_{D,C}"] & \mathcal{D}(D, GC) \\
\mathcal{C}(FD', C') \arrow[u, "\mathcal{C}(Ff,g)"] \arrow[r, "m_{D',C'}"'] & \mathcal{D}(D', GC') \arrow[u, "\mathcal{D}(f,Gg)"']
\end{tikzcd}$$

commutes in Set. Remind yourself that given $\alpha : FD' \to C'$, $\mathcal{C}(Ff, g)(\alpha) : FD \to C$ is the composite

$$FD \xrightarrow{Ff} FD' \xrightarrow{\alpha} C' \xrightarrow{g} C$$

Such a family $m = (m_{D,C} | D \in \mathcal{D}_0, C \in \mathcal{C}_0)$ is then completely determined by the values it takes on identities; i.e. the values

$$m_{D,FD}(\mathrm{id}_{FD}) : D \to GFD$$

For, given $\alpha : FD \to C$, since $\alpha = \mathcal{C}(\mathrm{id}_{FD}, \alpha)(\mathrm{id}_{FD})$,

$$m_{D,C}(\alpha) \quad = \quad m_{D,C}(\mathcal{C}(\mathrm{id}_{FD}, \alpha)(\mathrm{id}_{FD})) \quad = \quad \mathcal{D}(\mathrm{id}_D, G(\alpha))(m_{D,FD}(\mathrm{id}_{FD}))$$

which is the composite

$$D \xrightarrow{m_{D,FD}(\mathrm{id}_{FD})} GFD \xrightarrow{G(\alpha)} G(C)$$

The standard notation for $m_{D,FD}(\mathrm{id}_{FD})$ is $\eta_D : D \to GF(D)$.

Exercise 94 Show that $(\eta_D : D \in \mathcal{D}_0)$ is a natural transformation:

$$\mathrm{id}_{\mathcal{D}} \Rightarrow GF$$

By the same reasoning, the natural family $(m^{-1}_{D,C} | D \in \mathcal{D}_0, C \in \mathcal{C}_0)$ is completely determined by its actions on identities

$$m^{-1}_{GC,C}(\mathrm{id}_{GC}) : FGC \to C$$

Again, the family $(m^{-1}_{GC,C}(\mathrm{id}_{GC}) | C \in \mathcal{C}_0)$ is a natural transformation: $FG \Rightarrow \mathrm{id}_{\mathcal{C}}$. We denote its components by ε_C and this is also standard notation.

We have that $m^{-1}_{D,C}(\beta : D \to GC)$ is the composite

$$FD \xrightarrow{F\beta} FGC \xrightarrow{\varepsilon_C} C$$

Now making use of the fact that $m_{D,C}$ and $m^{-1}_{D,C}$ are each others inverse we get that for all $\alpha : FD \to C$ and $\beta : D \to GC$ the diagrams

$$\begin{tikzcd} D \arrow[r, "\beta"] \arrow[d, "\eta_D"'] & GC \\ GFD \arrow[r, "GF(\beta)"'] & GFG(C) \arrow[u, "G(\varepsilon_C)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FD \arrow[r, "\alpha"] \arrow[d, "F(\eta_D)"'] & C \\ FGFD \arrow[r, "FG(\alpha)"'] & FGC \arrow[u, "\varepsilon_C"'] \end{tikzcd}$$

commute; applying this to the identities on FD and GC we find that we have commuting diagrams of natural transformations:

$$\begin{tikzcd} G \arrow[r, Rightarrow, "\eta\star G"] \arrow[dr, Rightarrow, "\mathrm{id}_G"'] & GFG \arrow[d, Rightarrow, "G\circ\varepsilon"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, Rightarrow, "F\circ\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_F"'] & FGF \arrow[d, Rightarrow, "\varepsilon\star F"] \\ & F \end{tikzcd}$$

Here $\eta \star G$ denotes $(\eta_{GC} | C \in \mathcal{C}_0)$ and $G \circ \varepsilon$ denotes $(G(\varepsilon_C) | C \in \mathcal{C}_0)$.

Conversely, given $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ with natural transformations $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ which satisfy the above *triangle equalities*, we have that $F \dashv G$.

The tuple $(F, G, \varepsilon, \eta)$ is called an *adjunction*. η is the *unit* of the adjunction, ε the *counit*.

Exercise 95 Prove the statement above, that is: given $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ satisfying $(G \circ \varepsilon) \cdot (\eta \star G) = \mathrm{id}_G$ and $(\varepsilon \star F) \cdot (F \circ \eta) = \mathrm{id}_F$, we have $F \dashv G$.

Exercise 96 Given $\mathcal{C} \underset{G_1}{\overset{F_1}{\leftrightarrows}} \mathcal{D} \underset{G_2}{\overset{F_2}{\leftrightarrows}} \mathcal{E}$, if $F_1 \dashv G_1$ and $F_2 \dashv G_2$ then $F_1F_2 \dashv G_2G_1$.

Examples. The world is full of examples of adjoint functors. We have met several:

a) Consider the forgetful functor U : Grp $\to$ Set and the free functor F : Set $\to$ Grp. Given a function from a set A to a group G (which is an arrow $A \to U(G)$ in Set) we can uniquely extend it to a group homomorphism from $(\tilde{A}, \star)$ to G (see example e) of 1.1), i.e. an arrow $F(A) \to G$ in Grp, and conversely. This is natural in A and G, so $F \dashv U$;

b) The functor Dgrph $\to$ Cat given in example f) of 1.1 is left adjoint to the forgetful functor Cat $\to$ Dgrph;

c) Given functors $P \underset{G}{\overset{F}{\leftrightarrows}} Q$ between two preorders P and Q, $F \dashv G$ if and only if we have the equivalence

$$y \leq G(x) \Leftrightarrow F(y) \leq x$$

for $x \in P, y \in Q$; if and only if we have $FG(x) \leq x$ and $y \leq GF(y)$ for all x, y;

d) In example m) of 1.1 we did "abelianization" of a group G. We made use of the fact that any homomorphism $G \to H$ with H abelian, factors uniquely through $G/[G, G]$, giving a natural 1-1 correspondence

$$\mathrm{Grp}(G, I(H)) \xrightarrow{\sim} \mathrm{Abgp}(G/[G, G], H)$$

where I : Abgp $\to$ Grp is the inclusion. So abelianization is left adjoint to the inclusion functor of abelian groups into groups;

e) The *free monoid* $F(A)$ on a set A is just the set of strings on the alphabet A. F : Set $\to$ Mon is a functor left adjoint to the forgetful functor from Mon to Set;

f) Given a set X we have seen (example g) of 2.2) the product functor $(-) \times X$: Set $\to$ Set, assigning the product $Y \times X$ to a set Y.

Since there is a natural bijection between functions $Y \times X \to Z$ and functions $Y \to Z^X$, the functor $(-)^X$: Set $\to$ Set is right adjoint to $(-) \times X$;

g) Example e) of 2.2 gives two functors $F, G : \text{Set} \to \text{Cat}$. F and G are respectively left and right adjoint to the functor $\text{Cat} \overset{\text{Ob}}{\to} \text{Set}$ which assigns to a (small) category its set of objects (to be precise, for this example to work we have to take for Cat the category of *small* categories), and to a functor its action on objects.

h) Given a regular category $\mathcal{C}$ we saw in 4.1 that every arrow $f : X \to Y$ can be factored as a regular epi followed by a monomorphism. In Exercise 70 you were asked to show that there is a function $\exists_f : \text{SUB}(X) \to \text{Sub}(Y)$ such that the equivalence $\exists_f(M) \leq N \Leftrightarrow M \leq f^*(N)$ holds, for arbitrary subobjects M and N of X and Y, respectively.

But this just means thet $\exists_f \dashv f^*$

We can also express this logically: in the logic developed in chapter 4, for any formulas $M(x)$ and $N(y)$, the sequents

$$\exists x(f(x) = y \wedge M(x)) \vdash_y N(y)$$

and

$$M(x) \vdash_x N(f(x))$$

are equivalent.

One of the slogans of categorical logic is therefore, that "existential quantification is left adjoint to substitution".

i) Let $\mathcal{C}$ be a category with finite products; for $C \in \mathcal{C}_0$ consider the slice category $\mathcal{C}/C$. There is a functor $C^* : \mathcal{C} \to \mathcal{C}/C$ which assigns to D the object $C \times D \overset{\pi_C}{\to} C$ of $\mathcal{C}/C$, and to maps $D \overset{f}{\to} D'$ the map $\text{id}_C \times f$.

C^* has a left adjoint Σ_C which takes the domain: $\Sigma_C(D \to C) = D$.

j) Let $P : \text{Set}^{\text{op}} \to \text{Set}$ be the functor which takes the powerset on objects, and for $X \overset{f}{\to} Y$, $P(f) : P(Y) \to P(X)$ gives for each subset B of Y its inverse image under f.

Now P might as well be regarded as a functor $\text{Set} \to \text{Set}^{\text{op}}$; let's write $\bar{P}$ for that functor. Since there is a natural bijection:

$$\text{Set}(X, P(Y)) \overset{\sim}{\to} \text{Set}(Y, P(X)) = \text{Set}^{\text{op}}(\bar{P}(X), Y)$$

we have an adjunction $\bar{P} \dashv P$.

Exercise 97 A general converse to the last example. Suppose that $F : \text{Set}^{\text{op}} \to \text{Set}$ is a functor, such that for the corresponding functor $\bar{F} : \text{Set} \to \text{Set}^{\text{op}}$ we have that $\bar{F} \dashv F$. Then there is a set A such that F is naturally isomorphic to $\text{Set}(-, A)$.

Exercise 98 Suppose that $\mathcal{C} \xleftarrow{F} \mathcal{D}$ is a functor and that for each object C of $\mathcal{C}$ there is an object GC of $\mathcal{D}$ and an arrow $\varepsilon_C : FGC \to C$ with the property that for every object D of $\mathcal{D}$ and any map $FD \xrightarrow{f} C$, there is a unique $\tilde{f} : D \to GC$ such that

$$\begin{tikzcd} FD \arrow[rr, "f"] \arrow[dr, "F\tilde{f}"'] & & C \\ & FGC \arrow[ur, "\varepsilon_C"'] & \end{tikzcd}$$

commutes.

Prove that $G : \mathcal{C}_0 \to \mathcal{D}_0$ extends to a functor $G : \mathcal{C} \to \mathcal{D}$ which is right adjoint to F, and that $(\varepsilon_C : FGC \to C | C \in \mathcal{C}_0)$ is the counit of the adjunction.

Construct also the unit of the adjunction.

Exercise 99 Given $\mathcal{C} \xrightarrow{G} \mathcal{D}$, for each object D of $\mathcal{D}$ we let $(D{\downarrow}G)$ denote the category which has as objects pairs (C, g) where C is an object in $\mathcal{C}$ and $g : D \to GC$ is an arrow in $\mathcal{D}$. An arrow $(C, g) \to (C', g')$ in $(D{\downarrow}G)$ is an arrow $f : C \to C'$ in $\mathcal{C}$ which makes

$$\begin{tikzcd} & D \arrow[dl, "g"'] \arrow[dr, "g'"] & \\ GC \arrow[rr, "Gf"'] & & GC' \end{tikzcd}$$

commute.

Show that G has a left adjoint if and only if for each D, the category $(D{\downarrow}G)$ has an initial object.

Exercise 100 (Uniqueness of adjoints) Any two left (or right) adjoints of a given functor are isomorphic as objects of the appropriate functor category.

Exercise 101 $\mathcal{D} \to \mathbf{1}$ has a right adjoint iff $\mathcal{D}$ has a terminal object, and a left adjoint iff $\mathcal{D}$ has an initial object.

Exercise 102 Suppose $\mathcal{D}$ has both an initial and a terminal object; denote by L the functor $\mathcal{D} \to \mathcal{D}$ which sends everything to the initial, and by R the one which sends everything to the terminal object. Prove that $L \dashv R$.

Exercise 103 We are given an adjunction $\mathcal{E} \underset{I}{\overset{R}{\leftrightarrows}} \mathcal{S}$ with $R \dashv I$, unit η and counit ε.

a) Prove: I is faithful if and only if every component of ε is epi; and I is full if and only if every component of ε is split mono. Hint: use the fact that for an arrow $A \xrightarrow{f} B$ in $\mathcal{E}$, the composite arrow $RIA \xrightarrow{\varepsilon_A} A \xrightarrow{f} B$ transposes under the adjunction to the arrow $I(f) : I(A) \to I(B)$.

b) Now suppose I is full and faithful. Prove: if $F : \mathcal{A} \to \mathcal{E}$ is a diagram and IF has a limit in $\mathcal{S}$, then F has a limit in $\mathcal{E}$.

Exercise 104 Suppose that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories with pseudo inverse $G : \mathcal{D} \to \mathcal{C}$. Show that both $F \dashv G$ and $G \dashv F$ hold.

Exercise 105 Consider the situation of Exercise 93 and in particular the diagram given in part a) of that exercise.

a) Show that the functor F has a left adjoint L.

b) Show that this functor L does not preserve equalizers.

Exercise 106 An object M of a category $\mathcal{C}$ is called *injective* if for any diagram

$$\begin{tikzcd} A \arrow[d, "m"'] \arrow[dr, "f"] & \\ B & M \end{tikzcd}$$

with m a monomorphism, there exists an arrow $g : B \to M$ satisfying $gm = f$.

a) Let $\mathcal{C}, \mathcal{D}$ be categories and $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ functors with $F \dashv G$. Prove: if F preserves monos, then G preserves injective objects.

b) Formulate the statement dual to part a) (the dual notion of 'injective' is *projective*).

c) Now assume that in $\mathcal{C}$, for any object X there is an injective object M and a monomorphism $m : X \to M$ (one says: the category $\mathcal{C}$ *has enough injectives*). Prove the converse of part a).

5.2 Expressing (co)completeness by existence of adjoints; preservation of (co)limits by adjoint functors

Given categories $\mathcal{C}$ and $\mathcal{D}$, we defined for every functor $F : \mathcal{C} \to \mathcal{D}$ its *limit* (or limiting cone), if it existed, as a pair (D, μ) with $\mu : \Delta_D \Rightarrow F$, and (D, μ) terminal in the category of cones for F.

Any other natural transformation $\mu' : \Delta_{D'} \Rightarrow F$ factors uniquely through (D, μ) via an arrow $D' \to D$ in $\mathcal{D}$ and conversely, every arrow $D' \to D$ gives rise to a natural transformation $\mu' : \Delta_{D'} \Rightarrow F$.

So there is a 1-1 correspondence between

$$\mathcal{D}(D', D) \text{ and } \mathcal{D}^{\mathcal{C}}(\Delta_{D'}, F)$$

which is natural in D'.

Since every arrow $D' \to D''$ in $\mathcal{D}$ gives a natural transformation $\Delta_{D'} \Rightarrow \Delta_{D''}$ (example i) of 2.2), there is a functor $\Delta_{(-)} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$.

The above equation now means that:

Proposition 5.1 *$\mathcal{D}$ has all limits of type $\mathcal{C}$ (i.e. every functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ has a limiting cone in $\mathcal{D}$) if and only if $\Delta_{(-)}$ has a right adjoint.*

Exercise 107 Give an exact proof of this proposition.

Exercise 108 Use duality to deduce the dual of the proposition: $\mathcal{D}$ has all colimits of type $\mathcal{C}$ if and only if $\Delta_{(-)} : \mathcal{D} \to \mathcal{D}^{\mathcal{C}}$ has a left adjoint.

A very important aspect of adjoint functors is their behaviour with respect to limits and colimits.

Theorem 5.2 *Let $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ such that $F \dashv G$. Then:*

a) G preserves all limits which exist in $\mathcal{C}$;

b) F preserves all colimits which exist in $\mathcal{D}$.

Proof. Suppose $M : \mathcal{E} \to \mathcal{C}$ has a limiting cone (C, μ) in $\mathcal{C}$. Now a cone (D, ν) for GM is a natural family $D \xrightarrow{\nu_E} GM(E)$, i.e. such that

$$\begin{tikzcd} D \arrow[r, "\nu_E"] \arrow[dr, "\nu_{E'}"'] & GM(E) \arrow[d, "GM(e)"] \\ & GM(E') \end{tikzcd}$$

commutes for every $E \xrightarrow{e} E'$ in $\mathcal{E}$.

This transposes under the adjunction to a family $(FD \xrightarrow{\tilde{\nu}_E} ME | E \in \mathcal{E}_0)$ and the naturality requirement implies that

$$\begin{tikzcd} FD \arrow[r, "\tilde{\nu}_E"] \arrow[dr, "\tilde{\nu}_{E'}"'] & ME \arrow[d, "M(e)"] \\ & ME' \end{tikzcd}$$

commutes in $\mathcal{C}$, in other words, that (FD, ν) is a cone for M in $\mathcal{C}$. There is, therefore, a unique map of cones from $(FD, \tilde{\nu})$ to (C, μ).

Transposing back again, we get a unique map of cones $(D, \nu) \to (GC, G \circ \mu)$. That is to say that $(GC, G \circ \mu)$ is terminal in Cone(GM), so a limiting cone for GM, which was to be proved.

The argument for the other statement is dual. ∎

Exercise 109 Given $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, $F \dashv G$ and $M : \mathcal{E} \to \mathcal{C}$. Show that the functor Cone$(M) \to$ Cone(GM) induced by G has a left adjoint.

From the theorem on preservation of (co)limits by adjoint functors one can often conclude that certain functors cannot have a right or a left adjoint.

Examples

a) The forgetful functor Mon $\rightarrow$ Set does not preserve epis, as we have seen in 1.2. In chapter 3 we've seen that f is epi iff is a pushout; since left adjoints preserve identities and pushouts, they preserve epis; therefore the forgetful functor Mon $\rightarrow$ Set does not have a right adjoint;

b) The functor $(-) \times X$: Set $\rightarrow$ Set does not preserve the terminal object unless X is itself terminal in Set; therefore, it does not have a left adjoint for non-terminal X.

c) The forgetful functor Pos $\rightarrow$ Set has a left adjoint, but it cannot have a right adjoint: it preserves all coproducts, including the initial object, but not all coequalizers.

Exercise 110 Prove the last example. Hint: think of the coequalizer of the following two maps $\mathbb{Q} \rightarrow \mathbb{R}$: one is the inclusion, the other is the constant zero map.

Another use of the theorem has to do with the computation of limits. Many categories, as we have seen, have a forgetful functor to Set which has a left adjoint. So the forgetful functor preserves limits, and since these can easily be computed in Set, one already knows the "underlying set" of the vertex of the limiting cone one wants to compute.

Does a converse to the theorem hold? I.e. given $G : \mathcal{C} \rightarrow \mathcal{D}$ which preserves all limits; does G have a left adjoint? In general *no*, unless $\mathcal{C}$ is sufficiently complete, and a rather technical condition, the "solution set condition" holds. The *adjoint functor theorem* (Freyd) tells that in that case there is a converse:

Definition 5.3 (Solution set condition) *$G : \mathcal{C} \rightarrow \mathcal{D}$ satisfies the solution set condition (ssc) for an object D of $\mathcal{D}$, if there is a* set *X_D of objects of $\mathcal{C}$, such that every arrow $D \rightarrow GC$ factors as*

$$\begin{tikzcd} D \arrow[r] \arrow[dr] & GC \\ & GC' \arrow[u, "G(f)"'] \end{tikzcd}$$

for some $C' \in X_D$.

Theorem 5.4 (Adjoint Functor Theorem) *Let $\mathcal{C}$ be a locally small, complete category and $G : \mathcal{C} \rightarrow \mathcal{D}$ a functor. G has a left adjoint if and only if G preserves all small limits and satisfies the ssc for every object D of $\mathcal{D}$.*

Proof. I sketch the proof for the 'if' part; convince yourself that the 'only if' part is trivial.

For any object D of $\mathcal{D}$ let $D{\downarrow}G$ be the category defined in exercise 98. By that exercise, we are looking for an initial object of $D{\downarrow}G$.

The solution set condition means, that there is a set $\mathcal{K}_0$ of objects of $D{\downarrow}G$ such that for any object x of $D{\downarrow}G$ there is an element $k \in \mathcal{K}_0$ and an arrow $k \to x$ in $D{\downarrow}G$.

The fact that $\mathcal{C}$ is complete and that G preserves all small limits, entails that $D{\downarrow}G$ is complete. Moreover, $D{\downarrow}G$ is locally small as $\mathcal{C}$ is. Now let $\mathcal{K}$ be the full subcategory of $D{\downarrow}G$ with set of objects $\mathcal{K}_0$. Then since $D{\downarrow}G$ is locally small and $\mathcal{K}_0$ a set, $\mathcal{K}$ is small. Take, by completeness of $\mathcal{C}$, a vertex of a limiting cone for the inclusion: $\mathcal{K} \to D{\downarrow}G$. Call this vertex x_0. x_0 may not yet be an initial object of $D{\downarrow}G$, but now let $\mathcal{M}$ be the full subcategory of $D{\downarrow}G$ on the single object x_0 (M is a monoid), and let x be a vertex of a limiting cone for the inclusion $\mathcal{M} \to D{\downarrow}G$. x is the *joint equalizer* of all arrows $f : x_0 \to x_0$ in $D{\downarrow}G$, and this will be an initial object in $D{\downarrow}G$. ∎

Let me remark that in natural situations, the ssc is always satisfied. But then in those situations, one generally does not invoke the Adjoint Functor Theorem in order to conclude to the existence of a left adjoint. The value of this theorem is theoretical, rather than practical.

For small categories $\mathcal{C}$, the ssc is of course irrelevant. But categories which are small *and* complete are complete preorders, as we saw in chapter 3.

For preorders $\mathcal{C}$, $\mathcal{D}$ we have: if $\mathcal{C}$ is complete, then $G : \mathcal{C} \to \mathcal{D}$ has a left adjoint if and only if G preserves all limits, that is: greatest lower bounds $\bigwedge B$ for all $B \subseteq \mathcal{C}$. For, put

$$F(d) = \bigwedge\{c|d \leq G(c)\}$$

Then $F(d) \leq c'$ implies (since G preserves $\bigwedge$) $\bigwedge\{G(c)|d \leq G(c)\} \leq G(c')$ which implies $d \leq G(c')$ since $d \leq \bigwedge\{G(c)|d \leq G(c)\}$; conversely, $d \leq G(c')$ implies $c' \in \{c|d \leq G(c)\}$ so $F(d) = \bigwedge\{c|d \leq G(c)\} \leq c'$.

6 Monads and Algebras

Given an adjunction $(F, G, \varepsilon, \eta) : \mathcal{C} \rightleftarrows \mathcal{D}$ let us look at the functor $T = GF : \mathcal{D} \to \mathcal{D}$.

We have a natural transformation $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow T$ and a natural transformation $\mu : T^2 \Rightarrow T$. The components μ_D are

$$T^2(D) = GFGFD \stackrel{G(\varepsilon_{FD})}{\to} GFD = T(D)$$

Furthermore the equalities

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

hold. Here $(T\mu)_D = T(\mu_D) : T^3D \to TD$ and $(\mu T)_D = \mu_{TD} : T^3D \to TD$ (Similar for ηT and $T\eta$).

Exercise 111 Prove these equalities.

A triple (T, μ, η) satisfying these identities is called a *monad*. Try to see the formal analogy between the defining equalities for a monad and the axioms for a monoid: writing $m(e, f)$ for ef in a monoid, and η for the unit element, we have

$$\begin{array}{ll} m(e, m(g, h)) = m(m(e, g), h) & \text{(associativity)} \\ m(\eta, e) = m(e, \eta) = e & \text{(unit)} \end{array}$$

Following this one calls μ the *multiplication* of the monad, and η its *unit*.

Example. The powerset functor $\mathcal{P} : \mathrm{Set} \to \mathrm{Set}$ (example j) of 2.2, with η and μ indicated there) is a monad (check).

Dually, there is the notion of a *comonad* (L, δ, ε) on a category $\mathcal{C}$, with equalities

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta L"'] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "\mathrm{id}_L"'] \arrow[d, "\delta"] \arrow[dr, "\mathrm{id}_L"] & \\ L & L^2 \arrow[l, "\varepsilon L"] \arrow[r, "L\varepsilon"'] & L \end{tikzcd}$$

Given an adjunction $(F, G, \varepsilon, \eta)$, $(FG, \delta = F\eta G, \varepsilon)$ is a comonad on $\mathcal{C}$. We call δ the *comultiplication* and ε the *counit* (this is in harmony with the unit-counit terminology for adjunctions).

Although, in many contexts, comonads and the notions derived from them are at least as important as monads, the treatment is dual so I concentrate on monads.

Every adjunction gives rise to a monad; conversely, every monad arises from an adjunction, but in more than one way. Essentially, there are a maximal and a minimal solution to the problem of finding an adjunction from which a given monad arises.

Example. A monad on a poset P is a monotone function $T : P \to P$ with the properties $x \leq T(x)$ and $T^2(x) \leq T(x)$ for all $x \in P$; such an operation is also often called a *closure operation* on P. Note that $T^2 = T$ because T is monotone.

In this situation, let $Q \subseteq P$ be the image of T, with the ordering inherited from P. We have maps $r : P \to Q$ and $i : Q \to P$ such that ri is the identity on Q and $ir = T : P \to P$.

For $x \in P$, $y \in Q$ we have $x \leq i(y) \Leftrightarrow r(x) \leq y$ (check); so $r \dashv i$ and the operation T arises from this adjunction.

6.1 Algebras for a monad

Given a monad (T, η, μ) on a category $\mathcal{C}$, we define the category T-Alg of T-*algebras* as follows:

- *Objects* are pairs (X, h) where X is an object of $\mathcal{C}$ and $h : T(X) \to X$ is an arrow in $\mathcal{C}$ such that

$$\begin{tikzcd} T^2(X) \arrow[r, "T(h)"] \arrow[d, "\mu_X"'] & T(X) \arrow[d, "h"] \\ T(X) \arrow[r, "h"'] & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & T(X) \arrow[d, "h"] \\ & X \end{tikzcd}$$

  commute;

- *Morphisms*: $(X, h) \to (Y, k)$ are morphisms $X \xrightarrow{f} Y$ in $\mathcal{C}$ for which

$$\begin{tikzcd} T(X) \arrow[r, "T(f)"] \arrow[d, "h"'] & T(Y) \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  commutes.

Theorem 6.1 *There is an adjunction between T-Alg and $\mathcal{C}$ which brings about the given monad T.*

Proof. There is an obvious forgetful functor $U^T : T\text{-Alg} \to \mathcal{C}$ which takes (X, h) to X. I claim that U^T has a left adjoint F^T:

F^T assigns to an object X the T-algebra $T^2(X) \stackrel{\mu_X}{\to} T(X)$; to $X \stackrel{f}{\to} Y$ the map $T(f)$; this is an algebra map because of the naturality of μ. That $T^2(X) \stackrel{\mu_X}{\to} T(X)$ is an algebra follows from the defining axioms for a monad T.

Now given any arrow $g : X \to U^T(Y,h)$ we let $\tilde{g} : (T(X), \mu_X) \to (Y,h)$ be the arrow $T(X) \stackrel{T(g)}{\to} T(Y) \stackrel{h}{\to} Y$. This is a map of algebras since

$$\begin{tikzcd} T^2(X) \arrow[r, "T^2(g)"] \arrow[d, "\mu_X"'] & T^2(Y) \arrow[r, "T(h)"] \arrow[d, "\mu_Y"'] & T(Y) \arrow[d, "h"] \\ T(X) \arrow[r, "T(g)"'] & T(Y) \arrow[r, "h"'] & Y \end{tikzcd}$$

commutes; the left hand square is the naturality of μ; the right hand square is because (Y,h) is a T-algebra.

Conversely, given a map of algebras $f : (TX, \mu_X) \to (Y,h)$ we have an arrow $\bar{f} : X \to Y$ by taking the composite $X \stackrel{\eta_X}{\to} TX \stackrel{f}{\to} Y$.

Now $\tilde{\bar{f}} : TX \to Y$ is the composite

$$TX \stackrel{T(\eta_X)}{\to} T^2X \stackrel{T(f)}{\to} TY \stackrel{h}{\to} Y$$

Since f is a T-algebra map, this is

$$T(X) \stackrel{T(\eta_X)}{\to} T^2(X) \stackrel{\mu_X}{\to} T(X) \stackrel{f}{\to} Y$$

which is f by the monad laws.

Conversely, $\bar{\tilde{g}} : X \to Y$ is the composite

$$X \stackrel{\eta_X}{\to} TX \stackrel{T(g)}{\to} TY \stackrel{h}{\to} Y$$

By naturality of η and the fact that (Y,h) is a T-algebra, we conclude that $\bar{\tilde{g}} = g$. So we have a natural 1-1 correspondence

$$\mathcal{C}(X, U^T(Y,h)) \simeq T\text{-Alg}(F^T(X), (Y,h))$$

and our adjunction.

Note that the composite $U^T F^T$ is the functor T, and that the unit η of the adjunction is the unit of T; the proof that for the counit ε of $F^T \dashv U^T$ we have that

$$T^2 = U^T F^T U^T F^T \stackrel{U^T \varepsilon F^T}{\to} U^T F^T = T$$

is the original multiplication μ, is left to you. ∎

Exercise 112 Complete the proof.

Example. The *group monad*. Combining the forgetful functor $U : \text{Grp} \to \text{Set}$ with the left adjoint, the free functor $\text{Set} \to \text{Grp}$, we get the following monad on Set:

$T(A)$ is the set of sequences on the alphabet $A \sqcup A^{-1}$ (A^{-1} is the set $\{a^{-1} | a \in A\}$ of formal inverses of elements of A, as in example e) of 1.1) in which no

aa^{-1} or $a^{-1}a$ occur. The unit $A \stackrel{\eta_A}{\to} TA$ sends $a \in A$ to the string $\langle a \rangle$. The multiplication $\mu : T^2(A) \to T(A)$ works as follows. Define $(-)^- : A \sqcup A^{-1} \to A \sqcup A^{-1}$ by $a^- = a^{-1}$ and $(a^{-1})^- = a$. Define also $(-)^-$ on strings by $(a_1 \ldots a_n)^- = a_n^- \ldots a_1^-$. Now for an element of $TT(A)$, which is a string on the alphabet $T(A) \sqcup T(A)^{-1}$, say $\sigma_1 \ldots \sigma_n$, we let $\mu_A(\sigma_1 \ldots \sigma_n)$ be the concatenation of the strings $\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n$ on the alphabet $A \sqcup A^{-1}$, where $\tilde{\sigma}_i = \sigma_i$ if $\sigma_i \in T(A)$, and $\tilde{\sigma}_i = (\sigma_i)^-$ if $\sigma_i \in T(A)^{-1}$. Of course we still have to remove possible substrings of the form aa^{-1} etc.

Now let us look at algebras for the group monad: maps $T(A) \stackrel{h}{\to} A$ such that for a string of strings

$$\alpha = \sigma_1, \ldots, \sigma_n = \langle \langle s_1^1, \ldots, s_1^{k_1} \rangle, \ldots, \langle s_n^1, \ldots, s_n^{k_n} \rangle \rangle$$

we have that

$$h(\langle h\sigma_1, \ldots, h\sigma_n \rangle) = h(\langle s_1^1, \ldots, s_1^{k_1}, \ldots, s_n^1, \ldots, s_n^{k_n} \rangle)$$

and

$$h(\langle a \rangle) = a \text{ for } a \in A$$

I claim that this is the same thing as a group structure on A, with multiplication $a \cdot b = h(\langle a, b \rangle)$.

The unit element is given by $h(\langle\rangle)$; the inverse of $a \in A$ is $h(\langle a^{-1} \rangle)$ since

$$\begin{array}{rcl} h(\langle a, h(\langle a^{-1} \rangle) \rangle) & = & h(\langle h(\langle a \rangle), h(\langle a^{-1} \rangle) \rangle) = \\ h(\langle a, a^{-1} \rangle) & = & h(\langle\rangle), \text{ the unit element} \end{array}$$

Try to see for yourself how the associativity of the monad and its algebras transforms into associativity of the group law.

Exercise 113 Finish the proof of the theorem: for the group monad T, there is an equivalence of categories between T-Alg and Grp.

This situation is very important and has its own name:

Definition 6.2 *Given an adjunction* $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ *,* $F \dashv G$*, there is always a* comparison functor $K : \mathcal{C} \to T$*-Alg for* $T = GF$*, the monad induced by the adjunction.* K *sends an object* C *of* $\mathcal{C}$ *to the* T*-algebra* $GFG(C) \stackrel{G(\varepsilon_C)}{\to} G(C)$.

We say that the functor $G : \mathcal{C} \to \mathcal{D}$ *is* monadic*, or by abuse of language (if* G *is understood), that* $\mathcal{C}$ is monadic over $\mathcal{D}$*, if* K *is an equivalence.*

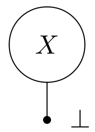
Exercise 114 Check that $K(C)$ is a T-algebra. Complete the definition of K as a functor. Check that in the example of the group monad, the functor T-Alg $\to$ Grp defined there is a pseudo inverse to the comparison functor K.

In many cases however, the situation is not monadic. Take the forgetful functor U : Pos $\to$ Set. It has a left adjoint F which sends a set X to the discrete ordering on X ($x \leq y$ iff $x = y$). Of course, UF is the identity on Set and the UF-algebras are just sets. The comparison functor K is the functor U, and this is not an equivalence.

Exercise 115 Why not?

Another example of a monadic situation is of importance in domain theory. Let $\mathrm{Pos}_\perp$ be the category of partially ordered sets with a least element, and order preserving maps which also preserve the least element.

There is an obvious inclusion functor $U : \mathrm{Pos}_\perp \to \mathrm{Pos}$, and U has a left adjoint F. Given a poset X, $F(X)$ is X "with a bottom element added":



Given $f : X \to Y$ in Pos, $F(f)$ sends the new bottom element of X to the new bottom element of Y, and is just f on the rest. If $f : X \to Y$ in Pos is a map and Y has a least element, we have $F(X) \to Y$ in $\mathrm{Pos}_\perp$ by sending $\perp$ to the least element of Y. So the adjunction is clear.

The monad $UF : \mathrm{Pos} \to \mathrm{Pos}$, just adding a least element, is called the *lifting monad.* Unit and multiplication are:

$\eta_X : X \to T(X)$ $\qquad\qquad$ $\mu_X : T^2(X) \to T(X)$

A T-algebra $h : TX \to X$ is first of all a monotone map, but since $h\eta_X = \mathrm{id}_X$, h is epi in Pos so surjective. It follows that X must have a least element $h(\perp)$. From the axioms for an algebra one deduces that h must be the identity when restricted to X, and $h(\perp)$ the least element of X.

Exercise 116 Given $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$, $F \dashv G$, $T = GF$. Prove that the comparison functor $K : \mathcal{C} \to T\text{-Alg}$ satisfies $U^T K = G$ and $KF = F^T$ where $T\text{-Alg} \underset{U^T}{\overset{F^T}{\rightleftarrows}} \mathcal{D}$ as in theorem 6.1.

Another poset example: algebras for the power set monad $\mathcal{P}$ on Set (example j) of 2.2). Such an algebra $h : \mathcal{P}(X) \to X$ must satisfy $h(\{x\}) = x$ and for $\alpha \subseteq \mathcal{P}(X)$:

$$h(\{h(A) | A \in \alpha\}) = h(\{x | \exists A \in \alpha(x \in A)\}) = h(\bigcup \alpha)$$

Now we can, given an algebra structure on X, define a partial order on X by putting $x \leq y$ iff $h(\{x,y\}) = y$.

Indeed, this is clearly reflexive and antisymmetric. As to transitivity, if $x \leq y$ and $y \leq z$ then

$$\begin{array}{rl} h(\{x,z\}) = & h(\{x, h(\{y,z\})\}) \quad = \\ h(\{h(\{x\}), h(\{y,z\})\}) = & h(\{x\} \cup \{y,z\}) \quad = \\ h(\{x,y\} \cup \{z\}) = & h(\{h(\{x,y\}), h(\{z\})\}) = \\ h(\{y,z\}) = & z \end{array}$$

so $x \leq z$.

Furthermore this partial order is *complete*: least upper bounds for arbitrary subsets exist. For $\bigvee B = h(B)$ for $B \subseteq X$: for $x \in B$ we have $h(\{x, h(B)\}) = h(\{x\} \cup B) = h(B)$ so $x \leq \bigvee B$; and if $x \leq y$ for all $x \in B$ then

$$\begin{array}{rl} h(\{h(B), y\}) = & h(B \cup \{y\}) \quad = \\ h(\bigcup_{x \in B} \{x,y\}) = & h(\{h(\{x,y\}) | x \in B\}) = \\ h(\{y\}) = & y \end{array}$$

so $\bigvee B \leq y$.

We can also check that a map of algebras is a $\bigvee$-preserving monotone function. Conversely, every $\bigvee$-preserving monotone function between complete posets determines a $\mathcal{P}$-algebra homomorphism.

We have an equivalence between the category of complete posets and $\bigvee$-preserving functions, and $\mathcal{P}$-algebras.

Exercise 117 Let $P : \mathrm{Set}^{\mathrm{op}} \to \mathrm{Set}$ be the contravariant powerset functor, and $\bar{P}$ its left adjoint, as in j) of 5.1. Let $T : \mathrm{Set} \to \mathrm{Set}$ the induced monad.

a) Describe unit and multiplication of this monad explicitly.

b) Show that $\mathrm{Set}^{\mathrm{op}}$ is equivalent to T-Alg [Hint: if this proves hard, have a look at VI.4.3 of Johnstone's "Stone Spaces"].

c) Conclude that there is an adjunction

$$\mathrm{CABool} \leftrightarrows \mathrm{Set}$$

which presents CABool as monadic over Set.

6.2 T-Algebras at least as complete as $\mathcal{D}$

Let T be a monad on $\mathcal{D}$. The following exercise is meant to show that if $\mathcal{D}$ has all limits of a certain type, so does T-Alg. In particular, if $\mathcal{D}$ is complete, so is T-Alg; this is often an important application of a monadic situation.

Exercise 118 Let $\mathcal{E}$ be a category such that every functor $M : \mathcal{E} \to \mathcal{D}$ has a limiting cone. Now suppose $M : \mathcal{E} \to T$-Alg. For objects E of $\mathcal{E}$, let $M(E)$ be the T-algebra $T(m_E) \stackrel{h_E}{\to} m_E$.

a) Let $(D, (\nu_E | E \in \mathcal{E}_0))$ be a limiting cone for $U^T M : \mathcal{E} \to \mathcal{D}$. Using the T-algebra structure on $M(E)$ and the fact that $U^T M(E) = m_E$, show that there is also a cone $(TD, (\pi_E | E \in \mathcal{E}_0))$ for $U^T M$;

b) Show that the unique map of cones: $(TD, \pi) \to (D, \nu)$ induces a T-algebra structure $TD \xrightarrow{h} D$ on D;

c) Show that $TD \xrightarrow{h} D$ is the vertex of a limiting cone for M in T-Alg.

This exercise gives a situation which has its own name. For a functor $G : \mathcal{C} \to \mathcal{D}$ we say that G *creates limits* of type $\mathcal{E}$ if for every functor $M : \mathcal{E} \to \mathcal{C}$ and every limiting cone (D, μ) for GM in $\mathcal{D}$, there is a unique cone (C, ν) for M in $\mathcal{C}$ which is taken by G to (D, μ), and moreover this unique cone is limiting for M in $\mathcal{C}$.

Clearly, if G creates limits of type $\mathcal{E}$ and $\mathcal{D}$ has all limits of type $\mathcal{E}$, then $\mathcal{C}$ has them, too. The exercise proves that the forgetful functor $U^T : T - \mathrm{Alg} \to \mathcal{D}$ creates limits of every type.

6.3 The Kleisli category of a monad

I said before that for a monad T on a category $\mathcal{D}$, there are a "maximal and a minimal solution" to the problem of finding an adjunction which induces the given monad.

We've seen the category T-Alg, which we now write as $\mathcal{D}^T$; we also write $G^T : T\text{-Alg} \to \mathcal{D}$ for the forgetful functor. In case T arises from an adjunction $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$, there was a comparison functor $\mathcal{C} \xrightarrow{K} \mathcal{D}^T$. In the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "K"] \arrow[dr, shift right, "G"'] & & \mathcal{D}^T \arrow[dl, shift left, "G^T"'] \\
& \mathcal{D} \arrow[ul, shift right, "F"'] \arrow[ur, shift left, "F^T"'] &
\end{tikzcd}$$

we have that $KF = F^T$ and $G^T K = G$.

Moreover, the functor K is unique with this property.

This leads us to define a category T-Adj of adjunctions $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ such that $GF = T$. A *map* of such T-adjunctions from $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ to $\mathcal{C}' \underset{G'}{\overset{F'}{\leftrightarrows}} \mathcal{D}$ is a functor $K : \mathcal{C} \to \mathcal{C}'$ satisfying $KF = F'$ and $G'K = G$.

What we have proved about T-Alg can be summarized by saying that the adjunction $\mathcal{D}^T \underset{G^T}{\overset{F^T}{\leftrightarrows}} \mathcal{D}$ is a *terminal object* in T-Adj. This was the "maximal" solution.

T-Adj has also an initial object: the *Kleisli category* of T, called $\mathcal{D}_T$. $\mathcal{D}_T$ has the same objects as $\mathcal{D}$, but a map in $\mathcal{D}_T$ from X to Y is an arrow $X \xrightarrow{f} T(Y)$

in $\mathcal{D}$. Composition is defined as follows: given $X \xrightarrow{f} T(Y)$ and $Y \xrightarrow{g} T(Z)$ in $\mathcal{D}$, considered as a composable pair of morphisms in $\mathcal{D}_T$, the composition gf in $\mathcal{D}_T$ is the composite

$$X \xrightarrow{f} T(Y) \xrightarrow{T(g)} T^2(Z) \xrightarrow{\mu_Z} T(Z)$$

in $\mathcal{D}$.

Exercise 119 Prove that composition is associative. What are the identities of $\mathcal{D}_T$?

The adjunction $\mathcal{D}_T \underset{G_T}{\overset{F_T}{\leftrightarrows}} \mathcal{D}$ is defined as follows: the functor G_T sends the object X to $T(X)$ and $f : X \to Y$ ($f : X \to T(Y)$ in $\mathcal{D}$) to

$$T(X) \xrightarrow{T(f)} T^2(Y) \xrightarrow{\mu_Y} T(Y)$$

The functor F_T is the identity on objects and sends $X \xrightarrow{f} Y$ to $X \xrightarrow{f} Y \xrightarrow{\eta_Y} T(Y)$, considered as $X \to Y$ in $\mathcal{D}_T$.

Exercise 120 Define unit and counit; check $F_T \dashv G_T$.

Exercise 121 Let T be a monad on $\mathcal{D}$. Call an object of T-Alg *free* if it is in the image of $F^T : \mathcal{D} \to T-\text{Alg}$. Show that the Kleisli category $\mathcal{D}_T$ is equivalent to the full subcategory of T-Alg on the free T-algebras.

Now for every adjunction $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ with $GF = T$, there is a unique comparison functor $L : \mathcal{D}_T \to \mathcal{C}$ such that $GL = G_T$ and $LF_T = F$.

L sends the object X to $F(X)$ and $f : X \to Y$ (so $f : X \to T(Y) = GF(Y)$ in $\mathcal{D}$) to its transpose $\tilde{f} : F(X) \to F(Y)$.

Exercise 122 Check the commutations. Prove the uniqueness of L w.r.t. these properties.

Exercise 123 Let Rng1 be the category of rings with unit and unitary ring homomorphisms. Since every ring with 1 is a (multiplicative) monoid, there is a forgetful functor $G : \text{Rng1} \to \text{Mon}$. For a monoid M, let $Z[M]$ be the ring of formal expressions

$$n_1c_1 + \cdots + n_kc_k$$

with $k \geq 0$, $n_1, \ldots, n_k \in Z$ and $c_1, \ldots, c_k \in M$. This is like a ring of polynomials, but multiplication uses the multiplication in M. Show that this defines a functor $F : \text{Mon} \to \text{Rng1}$ which is left adjoint to G, and that G is monadic, i.e. the category of GF-algebras is equivalent to Rng1. [Hint: Proceed as in the example of the powerset monad. That is, let $h : GF(M) \to M$ be a monoid homomorphism which gives M the structure of a GF-algebra. Find an abelian group structure on M such that M becomes a ring with unit]

Exercise 124 What does the Kleisli category for the covariant powerset monad look like?

7 Cartesian closed categories and the λ-calculus

Many set-theoretical constructions are completely determined (up to isomorphism, as always) by their categorical properties in Set. We are therefore tempted to generalize them to arbitrary categories, by taking the characteristic categorical property as a definition. Of course, this procedure is not really well-defined and it requires sometimes a real insight to pick the 'right' categorical generalization. For example, the category of sets has very special properties:

- $f : X \to Y$ is mono if and only if $fg = fh$ implies $g = h$ for any two maps $g, h : 1 \to X$, where 1 is a terminal object (we say 1 is *a generator*);
- objects X and Y are isomorphic if there exist monos $f : X \to Y$ and $g : Y \to X$ (the Cantor-Bernstein theorem);
- every mono $X \xrightarrow{f} Y$ is part of a coproduct diagram

$$\begin{tikzcd} X \arrow[dr, "f"] & \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

And if you believe the axiom of choice, there is its categorical version:

- Every epi is split

None of these properties is generally valid, and categorical generalizations based on them are usually of limited value.

In this chapter we focus on a categorical generalization of a set-theoretical concept which has proved to have numerous applications: Cartesian closed categories as the generalization of "function space".

In example f) of 5.1 we saw that the set of functions Z^X appears as the value at Z of the right adjoint to the product functor $(-) \times X$. A category is called *cartesian closed* if such right adjoints always exist. In such categories we may really think of this right adjoint as giving the "object of functions (or arrows)", as the treatment of the λ-calculus will make clear.

7.1 Cartesian closed categories (ccc's); examples and basic facts

Definition 7.1 *A category $\mathcal{C}$ is called* cartesian closed *or* a ccc *if it has finite products, and for every object X of $\mathcal{C}$ the product functor $(-) \times X$ has a right adjoint.*

Of course, "the" product functor only exists once we have chosen a product diagram for every pair of objects of $\mathcal{C}$. In this chapter we assume that we have such a choice, as well as a distinguished terminal object 1; and we assume also that for each object X we have a *specified* right adjoint to the functor $(-) \times X$, which we write as $(-)^X$ (Many authors write $X \Rightarrow (-)$, but I think that overloads the arrows notation too much). Objects of the form Z^X are called *exponents.*

We have the *unit*

$$Y \stackrel{\eta_{Y,X}}{\to} (Y \times X)^X$$

and *counit*

$$Y^X \times X \stackrel{\varepsilon_{Y,X}}{\to} Y$$

of the adjunction $(-) \times X \dashv (-)^X$. Anticipating the view of Y^X as the object of arrows $X \to Y$, we call ε *evaluation.*

Examples

a) A preorder (or partial order) is cartesian closed if it has a top element 1, binary meets $x \wedge y$ and for any two elements x, y an element $x \to y$ satisfying for each z:

$$z \leq x \to y \text{ iff } z \wedge x \leq y$$

b) Set is cartesian closed; Cat is cartesian closed (2.1);

c) Top is not cartesian closed. In chapter 4 it was remarked, that for non-locally compact spaces X, the functor $X \times (-)$ will not preserve quotients (coequalizers); hence, it cannot have a right adjoint. There are various subcategories of Top which are cartesian closed, if one takes as exponent Y^X the set of continuous maps $Y \to X$, topologized with the compact-open topology.

d) Pos is cartesian closed. The exponent Y^X is the set of all monotone maps $X \to Y$, ordered pointwise ($f \leq g$ iff for all $x \in X$, $fx \leq gx$ in Y);

e) Grp and Abgp are not cartesian closed. In both categories, the initial object is the one-element group. Since for non-initial groups G, $(-) \times G$ does not preserve the initial object, it cannot have a right adjoint (the same argument holds for Mon);

f) **1** is cartesian closed; **0** isn't (why?);

g) $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is cartesian closed. Products and 1 are given "pointwise" (in fact all limits are), that is $F \times G(C) = F(C) \times G(C)$ and $1(C)$ is the terminal 1 in Set, for all $C \in \mathcal{C}_0$.

   The construction of the exponent G^F is a nice application of the Yoneda lemma. Indeed, for G^F to be the right adjoint (at G) of $(-) \times F$, we need for every object C of $\mathcal{C}$:

$$\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C \times F, G) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C, G^F) \simeq G^F(C)$$

   where the last isomorphism is by the Yoneda lemma.

Now the assignment $C \mapsto \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(h_C \times F, G)$ defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, which we denote by G^F. At the same time, this construction defines a functor $(-)^F : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, which is right adjoint to $(-) \times F$. It is a nice exercise to prove this.

h) A monoid is never cartesian closed unless it is trivial. However, if from the definition of 'cartesian closed' one would delete the requirement that it has a terminal object, an interesting class of 'cartesian closed' monoids exists: the *C-mnoids* in the book "Higher Order Categorical Logic" by J. Lambek and Ph. Scott.

Exercise 125 Show that every Boolean algebra is cartesian closed (as a partial order).

Exercise 126 Show that CABool is not cartesian closed [use 2.3].

Exercise 127 Show that a complete partial order is cartesian closed if and only if it's a frame [see section 4.5].

Exercise 128 Let Ω be a frame. By the preceding exercise, it is cartesian closed; denote by $x{\to}y$ the exponent in Ω. This exercise is meant to let you show that the category $\mathcal{C}_\Omega$ from section 4.5 is cartesian closed.

a) Show that Ω also has greatest lower bounds $\bigwedge B$ for all subsets B.

b) Given objects (X, E_X) and (Y, E_Y), define their exponent $(Y, E_Y)^{(X,E_X)}$ as (Y^X, E) where Y^X is the set of all functions $X \to Y$ in Set, and

$$E(f) = \bigwedge\{E_X(x){\to}E_Y(f(x)) | x \in X\}$$

Show that this defines a right adjoint (at (Y, E_Y)) of $(-) \times (X, E_X)$.

Some useful facts:

- $\mathcal{C}$ is cartesian closed if and only if it has finite products, and for each pair of objects X, Y there is an object Y^X and an arrow $\varepsilon : Y^X \times X \to Y$ such that for every Z and map $Z \times X \xrightarrow{f} Y$ there is a unique $Z \xrightarrow{\tilde{f}} Y^X$ such that

$$\begin{tikzcd} Z \times X \arrow[rr, "f"] \arrow[dr, "\tilde{f} \times \mathrm{id}_X"'] & & Y \\ & Y^X \times X \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

commutes (use the result of exercise 98).

- In a ccc, there are natural isomorphisms $1^X \simeq 1$; $(Y \times Z)^X \simeq Y^X \times Z^X$; $(Y^Z)^X \simeq Y^{Z \times X}$.

- If a ccc has coproducts, we have $X \times (Y+Z) \simeq (X \times Y) + (X \times Z)$ and $Y^{Z+X} \simeq Y^Z \times Y^X$.

Exercise 129 Prove these facts.

Recall that two maps $Z \times X \to Y$ and $Z \to Y^X$ which correspond to each other under the adjunction are called each other's *transposes*.

Exercise 130 In a ccc, prove that the transpose of a composite $Z \xrightarrow{g} W \xrightarrow{f} Y^X$ is

$$Z \times X \xrightarrow{g \times \mathrm{id}_X} W \times X \xrightarrow{\tilde{f}} Y$$

if $\tilde{f}$ is the transpose of f.

Lemma 7.2 *In a ccc, given $f : X' \to X$ let $Y^f : Y^X \to Y^{X'}$ be the transpose of*

$$Y^X \times X' \xrightarrow{\mathrm{id} \times f} Y^X \times X \xrightarrow{\varepsilon} Y$$

Then for each $f : X' \to X$ and $g : Y \to Y'$ the diagram

$$\begin{tikzcd}
Y^X \arrow[r, "g^X"] \arrow[d, "Y^f"'] & Y'^X \arrow[d, "Y'^f"] \\
Y^{X'} \arrow[r, "g^{X'}"'] & Y'^{X'}
\end{tikzcd}$$

commutes.

Proof. By the exercise, the transposes of both composites are the top and bottom composites of the following diagram:

$$\begin{tikzcd}
 & Y'^X \times X' \arrow[r, "\mathrm{id} \times f"] & Y'^X \times X \arrow[dr, "\varepsilon"] & & \\
Y^X \times X' \arrow[ur, "g^X \times \mathrm{id}"] \arrow[r, "\mathrm{id} \times f"] \arrow[dr, "Y^f \times \mathrm{id}"'] & Y^X \times X \arrow[ur, "g^X \times \mathrm{id}"] \arrow[r, "\varepsilon"] & Y \arrow[r, "g"] & Y' \\
 & Y^{X'} \times X' \arrow[ur, "\varepsilon"] \arrow[r, "g^{X'} \times \mathrm{id}"'] & Y'^{X'} \times X' \arrow[ur, "\varepsilon"'] &
\end{tikzcd}$$

This diagram commutes because the right hand "squares" are naturality squares for ε, the lower left hand square commutes because both composites are the transpose of Y^f, and the upper left hand square commutes because both composites are $g^X \times f$. ∎

Proposition 7.3 *For every ccc $\mathcal{C}$ there is a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, assigning Y^X to (X, Y), and given $g : Y \to Y'$ and $f : X' \to X$, $g^f : Y^X \to Y'^{X'}$ is either of the composites in the lemma.*

Exercise 131 Prove the proposition.

7.2 Typed λ-calculus and cartesian closed categories

The λ-calculus is an extremely primitive formalism about functions. Basically, we can form functions (by λ-abstraction) and apply them to arguments; that's all. Here I treat briefly the *typed* λ-calculus.

We start with a set $\mathcal{S}$ of *type symbols* $S_1, S_2, \ldots$

Out of $\mathcal{S}$ we make the set of types as follows: every type symbol is a type, and if T_1 and T_2 are types then so is $(T_1{\Rightarrow}T_2)$.

We have also *terms* of each type (we label the terms like $t{:}T$ to indicate that t is a term of type T):

- we may have constants $c{:}T$ of type T;
- for every type T we have a denumerable set of variables $x_1{:}T, x_2{:}T, \ldots$;
- given a term $t{:}(T_1{\Rightarrow}T_2)$ and a term $s{:}T_1$, there is a term $(ts){:}T_2$;
- given $t{:}T_2$ and a variable $x{:}T_1$ there is a term $\lambda x.t{:}T_1{\Rightarrow}T_2$.

Terms $\lambda x.t$ are said to be formed by λ-*abstraction.* This procedure binds the variable x in t. Furthermore we have the usual notion of substitution for free variables in a term t (types have to match, of course). Terms of form (ts) are said to be formed by *application.*

In the λ-calculus, the only statements we can make are equality statements about terms. Again, I formulate the rules in terms of theories. First, let us say that a *language* consists of a set of type symbols and a set of constants, each of a type generated by the set of type symbols.

An *equality judgement* is an expression of the form $\Gamma|t = s{:}T$ (to be read: "Γ thinks that s and t are equal terms of type T "), where Γ is a finite set of variables which includes all the variables free in either t or s, and t and s are terms of type T.

A *theory* is then a set $\mathcal{T}$ of equality judgements which is closed under the following rules:

i) $\Gamma|t = s{:}T$ in $\mathcal{T}$ implies $\Delta|t = s{:}T$ in $\mathcal{T}$ for every superset Δ of Γ;

ii) $FV(t)|t = t{:}T$ is in $\mathcal{T}$ for every term $t{:}T$ of the language (again, $FV(t)$ is the set of free variables of t);
if $\Gamma|t = s{:}T$ and $\Gamma|s = u{:}T$ are in $\mathcal{T}$ then so is $\Gamma|t = u{:}T$;

iii) if $t(x_1, \ldots, x_n){:}T$ is a term of the language, with free variables $x_1{:}S_1,\ldots,x_n{:}S_n$, and $\Gamma|s_1 = t_1{:}S_1, \ldots, \Gamma|s_n = t_n{:}S_n$ are in $\mathcal{T}$ then

$$\Gamma|t[s_1/x_1, \ldots, s_n/x_n] = t[t_1/x_1, \ldots, t_n/x_n]{:}T$$

is in $\mathcal{T}$;

iv) if t and s are terms of type $(T_1{\Rightarrow}T_2)$, x a variable of type T_1 which does not occur in t or s, and $\Gamma \cup \{x\}|(tx) = (sx){:}T_2$ is in $\mathcal{T}$, then $\Gamma \setminus \{x\}|t = s{:}(T_1{\Rightarrow}T_2)$ is in $\mathcal{T}$;

v) if $s{:}T_1$ and $t{:}T_2$ are terms and x a variable of type T_2, then

$$FV(s) \setminus \{x\} \cup FV(t) | ((\lambda x.s)t) = s[t/x]{:}T_1$$

is in $\mathcal{T}$.

Given a language, an interpretation of it into a ccc $\mathcal{C}$ starts by choosing objects $[\![S]\!]$ of $\mathcal{C}$ for every type symbol S. This then generates objects $[\![T]\!]$ for every type T by the clause

$$[\![T_1 \Rightarrow T_2]\!] = [\![T_2]\!]^{[\![T_1]\!]}$$

The interpretation is completed by choosing interpretations

$$1 \xrightarrow{[\![c]\!]} [\![T]\!]$$

for every constant $c{:}T$ of the language.

Such an interpretation then generates, in much the same way as in chapter 4, interpretations of all terms. For a finite set $\Gamma = \{x_1{:}T_1, \ldots, x_n{:}T_n\}$ let's again write $[\![\Gamma]\!]$ for the product $[\![T_1]\!] \times \cdots \times [\![T_n]\!]$ (this is only defined modulo a permutation of the factors of the product, but that will cause us no trouble).

The interpretation of $t{:}T$ will now be an arrow

$$[\![FV(t)]\!] \xrightarrow{[\![t]\!]} [\![T]\!]$$

defined as follows:

- $[\![x]\!]$ is the identity on $[\![T]\!]$ for every variable $x{:}T$;
- given $[\![t]\!] : [\![FV(t)]\!] \to [\![T_2]\!]^{[\![T_1]\!]}$ and $[\![s]\!] : [\![FV(s)]\!] \to [\![T_1]\!]$ we let $[\![(ts)]\!] : [\![FV((ts))]\!] \to [\![T_2]\!]$ be the composite

  $$[\![FV((ts))]\!] \xrightarrow{\langle [\![t]\!]\pi_t, [\![s]\!]\pi_s \rangle} [\![T_2]\!]^{[\![T_1]\!]} \times [\![T_1]\!] \xrightarrow{\varepsilon} [\![T_2]\!]$$

  where π_t and π_s are the projections from $[\![FV((ts))]\!]$ to $[\![FV(t)]\!]$ and $[\![FV(s)]\!]$, respectively;
- given $[\![t]\!] : [\![FV(t)]\!] \to [\![T_2]\!]$ and the variable $x{:}T_1$ we let $[\![\lambda x.t]\!] : [\![FV(t) \setminus \{x\}]\!] \to [\![T_2]\!]^{[\![T_1]\!]}$ be the transpose of

  $$[\![FV(t) \setminus \{x\}]\!] \times [\![T_1]\!] \xrightarrow{\tilde{t}} [\![T_2]\!]$$

  where, if x occurs free in t so $[\![FV(t) \setminus \{x\}]\!] \times [\![T_1]\!] \simeq [\![FV(t)]\!]$, $\tilde{t}$ is just $[\![t]\!]$; and if x doesn't occur in t, $\tilde{t}$ is $[\![t]\!]$ composed with the obvious projection.

We now say that an equality judgement $\Gamma|t = s{:}T$ is *true* in this interpretation, if the diagram

$$\begin{tikzcd} & \llbracket FV(t) \rrbracket \arrow[dr, "\llbracket t \rrbracket"] & \\ \llbracket \Gamma \rrbracket \arrow[ur, "\pi_t"] \arrow[dr, "\pi_s"'] & & \llbracket T \rrbracket \\ & \llbracket FV(s) \rrbracket \arrow[ur, "\llbracket s \rrbracket"'] & \end{tikzcd}$$

commutes (again, π_s and π_t projections).

Lemma 7.4 *Let $t(x_1, \ldots, x_n){:}T$ have free variables $x_i{:}T_i$ and let $t_i{:}T_i$ be terms. Let*

$$\tilde{t}_i : \llbracket FV(t[t_1/x_1, \ldots, t_n/x_n]) \rrbracket \to \llbracket T_i \rrbracket$$

be the obvious composite of projection and $\llbracket t_i \rrbracket$.

Then the composition

$$\llbracket FV(t[t_1/x_1, \ldots, t_n/x_n]) \rrbracket \stackrel{\langle \tilde{t}_i | i=1,..n \rangle}{\longrightarrow} \prod_{i=1}^{n} \llbracket T_i \rrbracket = \llbracket FV(t) \rrbracket \stackrel{\llbracket t \rrbracket}{\to} \llbracket T \rrbracket$$

is the interpretation $\llbracket t[t_1/x_1, \ldots, t_n/x_n] \rrbracket$.

Exercise 132 Prove the lemma [take your time. This is not immediate].

Theorem 7.5 *Let S be a set of equality judgements and $\mathcal{T} = Cn(S)$ be the least theory containing S. If every judgement of S is true in the interpretation, so is every judgement in $\mathcal{T}$.*

Proof. Again, we show that the set of true judgements is a theory, i.e. closed under the rules in the definition of a theory.
i) and ii) are trivial;
iii) follows at once by lemma 7.4;
iv) Since $\Gamma \cup \{x\} = (\Gamma \setminus \{x\}) \cup \{x\}$ and because of the inductive hypothesis, we have that

$$\begin{tikzcd} & \llbracket FV(s) \rrbracket \times \llbracket T_1 \rrbracket \arrow[r, "\llbracket s \rrbracket \times \mathrm{id}"'] & \llbracket T_1 \Rightarrow T_2 \rrbracket \times \llbracket T_1 \rrbracket \arrow[dr, "\varepsilon"] & \\ \llbracket \Gamma \setminus \{x\} \cup \{x\} \rrbracket \arrow[ur, "\pi_s \times \mathrm{id}"] \arrow[dr, "\pi_t \times \mathrm{id}"'] & & & \llbracket T_2 \rrbracket \\ & \llbracket FV(t) \rrbracket \times \llbracket T_1 \rrbracket \arrow[r, "\llbracket t \rrbracket \times \mathrm{id}"'] & \llbracket T_1 \Rightarrow T_2 \rrbracket \times \llbracket T_1 \rrbracket \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

commutes. Taking the transposes of both maps, we get the equality we want.
v) According to lemma 7.4, $[\![FV(s[t/x])]\!] \stackrel{[\![s[t/x]]\!]}{\longrightarrow} [\![T_1]\!]$ is

$$[\![FV(s[t/x])]\!] \stackrel{\tilde{t}}{\to} [\![FV(s)]\!] \stackrel{[\![s]\!]}{\to} [\![T_1]\!]$$

This is the same as

$$[\![FV(s[t/x])]\!] \stackrel{\langle \pi, [\![t]\!] \rangle}{\to} [\![FV(s) \setminus \{x\}]\!] \times [\![T_2]\!] \stackrel{[\![\lambda x.s]\!] \times \mathrm{id}}{\longrightarrow} [\![T_2 \Rightarrow T_1]\!] \times [\![T_2]\!] \stackrel{\varepsilon}{\to} [\![T_1]\!]$$

which is

$$[\![FV((\lambda x.s)t)]\!] \stackrel{[\![((\lambda x.s)t)]\!]}{\longrightarrow} [\![T_1]\!]$$

∎

There is also a *completeness theorem*: if a judgement $\Gamma | t = s{:}T$ is true in all possible interpretations, then every theory (in a language this judgement is in) contains it.

The relevant construction is that of a syntactic cartesian closed category out of a theory, and an interpretation into it which makes exactly true the judgements in the theory. The curious reader can find the, somewhat laborious, treatment in Lambek & Scott's "Higher Order Categorical Logic".

7.3 Representation of primitive recursive functions in ccc's with natural numbers object

Dedekind observed, that in Set, the set ω is characterized by the following property: given any set X, any element $x \in X$ and any function $X \stackrel{f}{\to} X$, there is a unique function $F : \omega \to X$ such that $F(0) = x$ and $F(x+1) = f(F(x))$.

Lawvere took this up, and proposed this *categorical* property as a definition (in a more general context) of a "natural numbers object" in a category.

Definition 7.6 *In a category $\mathcal{C}$ with terminal object 1, a* natural numbers object *is a triple $(0, N, S)$ where N is an object of $\mathcal{C}$ and $1 \stackrel{0}{\to} N$, $N \stackrel{S}{\to} N$ arrows in $\mathcal{C}$, such that for any other such diagram*

$$1 \xrightarrow{x} X \xrightarrow{f} X$$

there is a unique *map $\phi : N \to X$ making*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "S"] \arrow[d, "\phi"'] & N \arrow[d, "\phi"] \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

commute.

Of course we think of 0 as the zero element, and of S as the successor map. The defining property of a natural numbers object enables one to "do recursion", a weak version of which we show here: we show that every primitive recursive function can be represented in a ccc with natural numbers object.

Definition 7.7 *Let $\mathcal{C}$ be a ccc with natural numbers object $(0, N, S)$. We say that a number-theoretic function $F : \mathbb{N}^k \to \mathbb{N}$ is* represented *by an arrow $f : N^k \to N$ if for any k-tuple of natural numbers $n_1, \ldots n_k$, the diagram*

$$\begin{tikzcd} 1 \arrow[r, "0"] \arrow[dr, "0"'] & N \arrow[r, "\langle S^{n_1}, \ldots, S^{n_k} \rangle"] & N^k \arrow[d, "f"] \\ & N \arrow[r, "S^{F(n_1,\ldots,n_k)}"'] & N \end{tikzcd}$$

commutes.

Recall that the class of *primitive recursive* functions is given by the following clauses:

- The constant zero function $\lambda\vec{x}.0 : \mathbb{N}^k \to \mathbb{N}$, the function $\lambda x.x + 1 : \mathbb{N} \to \mathbb{N}$ and the projections $\lambda\vec{x}.x_i : \mathbb{N} \to \mathbb{N}$ are primitive recursive;
- The primitive recursive functions are closed under composition: if $F_1, \ldots, F_k : \mathbb{N}^l \to \mathbb{N}$ and $G : \mathbb{N}^k \to \mathbb{N}$ are primitive recursive, then so is $G(\langle F_1, \ldots, F_k \rangle) : \mathbb{N}^l \to \mathbb{N}$;
- The primitive recursive functions are closed under *definition by primitive recursion*: if $G : \mathbb{N}^k \to \mathbb{N}$ and $H : \mathbb{N}^{k+2} \to \mathbb{N}$ are primitive recursive, and $F : \mathbb{N}^{k+1} \to \mathbb{N}$ is defined by $F(0, \vec{x}) = G(\vec{x})$ and $F(n+1, \vec{x}) = H(n, F(n, \vec{x}), \vec{x})$ then F is primitive recursive.

Proposition 7.8 *In a ccc $\mathcal{C}$ with natural numbers object, every primitive recursive function is representable.*

Proof. I do only the case for definition by primitive recursion. So by inductive hypothesis we have arrows G and H representing the homonymous functions. By interpretation of the λ-calculus, I use λ-terms: so there is an arrow

$$\lambda\vec{x}.G(\vec{x}) : 1 \to N^{(N^k)}$$

and an arrow

$$\lambda\vec{x}.H(n, \phi(\vec{x}), \vec{x}) : N^{(N^k)} \times N \to N^{(N^k)}$$

which is the interpretation of a term with free variables $\phi{:}N^{(N^k)}$ and $n{:}N$; this map is the exponential transpose of the map which intuitively sends $(n, \phi, \vec{x})$ to $(n, \phi(\vec{x}), \vec{x})$. Now look at

$$1 \xrightarrow{\langle \lambda\vec{x}.G(\vec{x}), 0 \rangle} N^{(N^k)} \times N \xrightarrow{(\lambda\vec{x}.H(n,\phi(\vec{x}),\vec{x})) \times S} N^{(N^k)} \times N$$

By the natural numbers object property, there is now a unique map

$$\bar{F} = \langle \tilde{F}, \sigma \rangle : N \to N^{(N^k)} \times N$$

which makes the following diagram commute:

$$\begin{tikzcd}
1 \arrow[r, "0"] \arrow[dr, "\langle \lambda \vec{x}.G(\vec{x}),0\rangle"'] & N \arrow[r, "S"] \arrow[d, "\bar{F}"] & N \arrow[d, "\bar{F}"] \\
& N^{(N^k)} \times N \arrow[r, "(\lambda \vec{x}.H(n,\phi(\vec{x}),\vec{x}))\times S"'] & N^{(N^k)} \times N
\end{tikzcd}$$

One verifies that σ is the identity, and that the composite

$$N^{k+1} \xrightarrow{\tilde{F}\times \mathrm{id}} N^{(N^k)} \times N^k \xrightarrow{\varepsilon} N$$

represents F. ∎

Exercise 133 Make these verifications.

One could ask: what is the class of those numerical functions (that is, functions $\mathbb{N}^k \to \mathbb{N}$) that are representable in every ccc with natural numbers object? It is not hard to see, that there are representable functions which are not primitive recursive (for example, the Ackermann function). On the other hand, Logic teaches us that every such representable function must be recursive, and that there are recursive, non-representable functions.

The answer is: the representable functions are precisely the so-called ε_0-*recursive functions* from Proof Theory; and this was essentially shown by Gödel in 1958.

8 Recursive Domain Equations

A *recursive domain equation* in a category $\mathcal{C}$ is an "equation" of the form:

$$X \cong F(X, \ldots, X)$$

where F is a functor: $(\mathcal{C}^{\mathrm{op}})^n \times \mathcal{C}^m \to \mathcal{C}$.

Often, we are interested in not just any solution of such an equation, but in certain 'universal' solutions. Consider, as an example, the case $\mathcal{C} = \mathrm{Set}$, and $F(X) = 1 + (A \times X)$ for a fixed, nonempty set A. There are many solutions of $X \cong F(X)$ but one stands out: it is the set of *finite sequences of elements of* A.

In what sense is this a universal solution? Do such solutions always exist?

There is a piece of theory about this, which is by now a classic in theoretical Computer Science and was developed by Dana Scott around 1970; it is concerned with a certain subcategory of Pos. It is a nice application of the methods of category theory.

8.1 The category $\mathbb{CPO}$

Let $(P, \leq)$ be a partially ordered set. A *downset* or *downwards closed subset* of P is a subset $A \subseteq P$ such that if $a \in A$ and $p \leq a$, then $p \in A$. The *downwards closure* $\downarrow A$ of $A \subseteq P$ is the least downset of P containing A: $\downarrow A = \{p \in P \mid \exists a \in A. p \leq a\}$. We write $\downarrow p$ for $\downarrow\{p\}$.

An ω-*chain* in P is a function $f : \mathbb{N} \to P$ such that $f(0) \leq f(1) \leq \ldots$.
$(P, \leq)$ is a *cpo* or ω-*complete partial order* if every ω-chain in P has a colimit (i.e., least upper bound). This colimit is denoted $\bigsqcup_{n \in \mathbb{N}} f(n)$.

A monotone function $f : P \to Q$ between cpo's is *continuous* if it preserves least upper bounds of ω-chains.

Exercise 134 Every cpo P can be regarded as a topological space, in the following way: open sets are those sets $A \subseteq P$ which are *upwards closed* ($a \in A \wedge a \leq b \Rightarrow b \in A$) and such that for any chain $f : \mathbb{N} \to P$, if $\bigsqcup_{n \in \mathbb{N}} f(n) \in A$ then $f(n) \in A$ for some $n \in \mathbb{N}$. Show that $f : P \to Q$ is continuous if and only if f is continuous w.r.t. the topology just defined.

There is a category $\mathbb{CPO}$ of cpo's and continuous maps, and this category is our object of study for a while. Since every continuous function is monotone, there is a forgetful functor $U : \mathbb{CPO} \to \mathrm{Pos}$.

Theorem 8.1 $U : \mathbb{CPO} \to \mathrm{Pos}$ *is monadic.*

Proof. We have to show that U has a left adjoint $F : \mathrm{Pos} \to \mathbb{CPO}$ such that $\mathbb{CPO}$ is equivalent to the category of UF-algebras on Pos.

Call a subset A of a poset P an ω-*ideal* if there is an ω-chain $f : \mathbb{N} \to P$ such that A is the downwards closure of the image of f. Let $\omega{-}\mathrm{Idl}(P)$ the set of ω-ideals of P, ordered by inclusion. If $\varphi : P \to Q$ is a monotone map and

$A \subseteq P$ an ω-ideal, then $\downarrow\varphi[A] = \{q \in Q \mid \exists a \in A.q \leq \varphi(a)\}$ is also an ω-ideal of Q, for if $A = \downarrow\mathrm{im}(f)$ for $f : \mathbb{N} \to P$ then $\downarrow\varphi[A] = \downarrow\mathrm{im}(\varphi \circ f)$.

If $A_0 \subseteq A_1 \subseteq \ldots$ is an ω-chain of elements of ω−Idl(P) then also $\bigcup_{n\in\mathbb{N}} A_n \in \omega$−Idl$(P)$, for, if $A_i = \downarrow\mathrm{im}(f_i)$ define $f : \mathbb{N} \to P$ by:

$$f(n) = \begin{array}{l} f_n(m) \text{ where } m \text{ is minimal such that} \\ f_n(m) \text{ is an upper bound of} \\ \{f_i(k) \mid i, k \in \{0, \ldots, n\}\} \cup \{f(k) \mid k < n\} \end{array}$$

Then f is a chain and $\bigcup_{n\in\mathbb{N}} A_n = \downarrow\mathrm{im}(f)$.

So ω−Idl(P) is a cpo; and since (for a monotone $\varphi : P \to Q$) the map $A \mapsto \downarrow\varphi[A]$ commutes with unions of ω-chains, it is a continuous map: ω−Idl$(P) \to \omega$−Idl(Q). So we have a functor $F :$ Pos $\to \mathbb{CPO}$: $F(P) = \omega$−Idl(P), and for $\varphi : P \to Q$ in Pos, $F(\varphi) : \omega$−Idl$(P) \to \omega$−Idl(Q) is the map which sends A to $\downarrow\varphi[A]$.

Every monotone function $f : P \to U(Q)$ where Q is a cpo, gives a continuous function $\tilde{f} : \omega$−Idl$(P) \to Q$ defined as follows: given $A \in \omega$−Idl(P), if $A = \downarrow\mathrm{im}(g)$ for a chain $g : \mathbb{N} \to P$, let $\tilde{f}(A)$ be the least upper bound in Q of the chain $f \circ g$. This is independent of the choice of g, for if $\downarrow\mathrm{im}(g) = \downarrow\mathrm{im}(g')$ then the chains $f \circ g$ and $f \circ g'$ have the same least upper bound in Q.

In the other direction, first let $\eta_P : P \to \omega$−Idl(P) be defined by $\eta_P(p) = \downarrow p$. Every continuous function $g : \omega$−Idl$(P) \to Q$ gives a monotone $\bar{g} : P \to U(Q)$ by composition with η_P.

Exercise 135 Check that these two operations define a natural 1-1 correspondence between Pos$(P, U(Q))$ and $\mathbb{CPO}(\omega$−Idl$(P), Q)$ and therefore an adjunction $F \dashv U$ of which $\eta = (\eta_P)_P$ is the unit. What is the counit of this adjunction? Is it iso? Epi? What do you conclude about the functor U?

The monad UF has η as unit, and as multiplication

$$\mu = \bigcup : \omega\text{−Idl}(\omega\text{−Idl}(P)) \to \omega\text{−Idl}(P)$$

taking the union of an ω-ideal of ω-ideals of P.

Exercise 136 Check that μ is well-defined. Prove that every UF-algebra is a cpo [Hint: compare with the proof that algebras for the powerset monad are equivalent to join-complete posets and join-preserving maps], so that $\mathbb{CPO}$ is equivalent to UF-Alg.

■

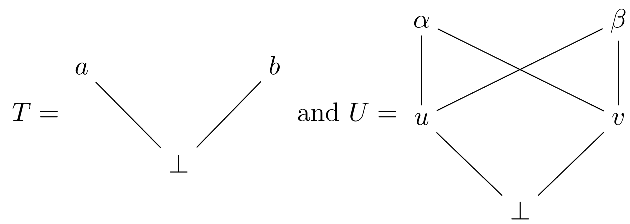
A corollary is now that $\mathbb{CPO}$ has all the limits that Pos has (that is, all small limits), and that these are created by the forgetful functor U. So, the limit in Pos of a diagram of cpo's and continuous maps, is also a cpo.

We shall also consider the category $\mathbb{CPO}_\perp$ of cpo's with a least element. $\mathbb{CPO}_\perp$ is a full subcategory of $\mathbb{CPO}$ (i.e., maps between objects of $\mathbb{CPO}_\perp$ are continuous but don't have to preserve the least element).

It is, categorically speaking, bad practice to require properties of objects without requiring the maps to preserve them. This is borne out by the fact that $\mathbb{CPO}_\perp$ loses the nice properties of $\mathbb{CPO}$:

Fact. $\mathbb{CPO}_\perp$ is neither finitely complete nor finitely cocomplete.

For instance consider the cpo's:



Both T and U are objects of $\mathbb{CPO}_\perp$. Let $f, g : T \to U$ be defined by: $f(a) = g(a) = \alpha$, $f(b) = g(b) = \beta$, $f(\perp) = u$, $g(\perp) = v$. f and g are maps of $\mathbb{CPO}_\perp$, but cannot have an equalizer in $\mathbb{CPO}_\perp$.

Exercise 137 Prove this. Prove also that the coproduct of two one-element cpo's cannot exist in $\mathbb{CPO}_\perp$.

A map of cpo's with least elements which preserves the least element is called *strict*. The category $(\mathbb{CPO}_\perp)_s$ of cpo's with least element and strict continuous maps, is monadic over $\mathbb{CPO}$ by the "lifting monad": adding a least element (see chapter 6), and therefore complete.

Lemma 8.2 *Let P be a cpo and $(x_{ij})_{i,j\in\mathbb{N}}$ be a doubly indexed set of elements of P such that $i \leq i'$ and $j \leq j'$ implies $x_{ij} \leq x_{i'j'}$. Then*

$$\bigsqcup_{i\in\mathbb{N}} \bigsqcup_{j\in\mathbb{N}} x_{ij} = \bigsqcup_{j\in\mathbb{N}} \bigsqcup_{i\in\mathbb{N}} x_{ij} = \bigsqcup_{i\in\mathbb{N}} x_{ii}$$

Exercise 138 Prove lemma 8.2.

Theorem 8.3 $\mathbb{CPO}$ *is cartesian closed.*

Proof. The exponent P^Q of two cpo's is the set of continuous maps from Q to P, ordered pointwise (i.e. $f \leq g$ iff $\forall q \in Q. f(q) \leq g(q)$). This is a cpo, because given a chain $f_0 \leq f_1 \leq \ldots$ of continuous maps, taking least upper bounds pointwise yields a continuous map:

$$f(q) = \bigsqcup_{i\in\mathbb{N}} f_i(q)$$

For, using lemma 8.2, $f(\bigsqcup_j q_j) = \bigsqcup_i \bigsqcup_j f_i(q_j) = \bigsqcup_j \bigsqcup_i f_i(q_j) = \bigsqcup_j f(q_j)$ ∎

Theorem 8.4 *Let P be a cpo with least element $\perp$. Then:*

a) *Every continuous map $f : P \to P$ has a least fixed point* $\mathrm{fix}(f)$ *(i.e. a least x with $f(x) = x$);*

b) *The assignment $f \mapsto \mathrm{fix}(f)$ is a continuous function: $P^P \to P$.*

Proof. Consider the chain $\perp \leq f(\perp) \leq f^2(\perp) \leq \ldots$. It's a chain because f is monotone. Let a be its least upper bound in P. Since f is continuous, $f(a) = f(\bigsqcup_{n\in\mathbb{N}} f^n(\perp)) = \bigsqcup_{n\in\mathbb{N}} f^{n+1}(\perp) = a$, so a is a fixed point; if b is another fixed point of f then since $\perp \leq b = f(b)$, $f^n(\perp) \leq b$ for all n, hence $a \leq b$, so a is the least fixed point of f.

For the second statement, first notice that if $f \leq g$ in P^P then $f^n(\perp) \leq g^n(\perp)$ so $\mathrm{fix}(f) \leq \mathrm{fix}(g)$, so fix is monotone; and if $f_1 \leq f_2 \leq \ldots$ then $\mathrm{fix}(\bigsqcup_n f_n) = \bigsqcup_i (\bigsqcup_n f_n)^i(\perp) \geq \bigsqcup_i \bigsqcup_n f_n^i(\perp) =$ (by lemma 8.2) $\bigsqcup_n \bigsqcup_i f_n^i(\perp) = \bigsqcup_n \mathrm{fix}(f_n)$. The other inequality follows from the monotonicity of fix. ∎

For purposes of interpretation of recursion equations, it is convenient to have a notation for fixed points of functions of more than one variable. Let P and Q be cpo's with $\perp$ and $f : P \times Q \to P$ continuous; by cartesian closedness of $\mathbb{CPO}$ we have $\tilde{f} : Q \to P^P$ and we can consider the composite

$$Q \stackrel{\tilde{f}}{\to} P^P \stackrel{\mathrm{fix}}{\to} P$$

which is a continuous map by theorem 8.4. $\mathrm{fix}(\tilde{f}(q))$ is the least fixed point of the function which sends p to $f(p, q)$; we write $\mu p.f(p, q)$ for this.

Békic' theorem[1] says that if we want to find a *simultaneous* fixed point $(x, y) \in P \times Q$ of a function $f : P \times Q \to P \times Q$, we can do it in two steps, each involving a single fixed point calculation:

Theorem 8.5 (Békic's simultaneous fixed point theorem) *Let P and Q be cpo's with $\perp$ and $f : P \times Q \to P$, $g : P \times Q \to Q$ be continuous maps. Then the least fixed point of the map $\langle f, g\rangle : P \times Q \to P \times Q$ is the pair $(\hat{x}, \hat{y}) \in P \times Q$, where*

$$\begin{aligned} \hat{x} &= \mu x.f(x, \mu y.g(x, y)) \\ \hat{y} &= \mu y.g(\hat{x}, y) \end{aligned}$$

Proof. The least fixed point a of a function f has the property, that for any y, if $f(y) \leq y$ then $a \leq y$ (check this!), and moreover it is *characterized* by this property. Therefore the element $\mu x.\Phi(x, \vec{y})$ satisfies the *rule*:

$$\Phi(x', \vec{y}) \leq x' \Rightarrow \mu x.\Phi(x, \vec{y}) \leq x'$$

and is characterized by it. Now suppose:

$$\begin{array}{ll} (1) & f(a, b) \leq a \\ (2) & g(a, b) \leq b \end{array}$$

From (2) and the rule we get $\mu y.g(a, y) \leq b$, hence by (1) and monotonicity of f, $f(a, \mu y.g(a, y)) \leq f(a, b) \leq a$. Applying the rule again yields:

$$\hat{x} = \mu x.f(a, \mu y.g(x, y)) \leq a$$

[1]This theorem is known in recursion theory as *Smullyan's double recursion theorem*

so by (2) and monotonicity of g: $g(\hat{x}, b) \leq g(a, b) \leq b$, so by the rule, $\mu y.g(\hat{x}, y) \leq b$. We have derived that $(\hat{x}, \hat{y}) \leq (a, b)$ from the assumption that $\langle f, g\rangle(a, b) \leq (a, b)$; this characterizes the least fixed point of $\langle f, g\rangle$, which is therefore $(\hat{x}, \hat{y})$. ∎

Exercise 139 Generalize theorem 8.5 to 3 continuous functions with 3 variables, and, if you have the courage, to n continuous functions with n variables.

Exercise 140 Suppose D, E are cpo's with $\perp$ and $f : D \to E$, $g : E \to D$ continuous. Show that $\mu d.gf(d) = g(\mu e.fg(e))$ [Hint: use the rule given in the proof of theorem 8.5]

8.2 The category of cpo's with $\perp$ and embedding-projection pairs; limit-colimit coincidence; recursive domain equations

So far, we have seen that the category $\mathbb{CPO}$ is cartesian closed and complete. About $\mathbb{CPO}_\perp$ we can say that:

- $\mathbb{CPO}_\perp$ has products and the inclusion $\mathbb{CPO}_\perp \to \mathbb{CPO}$ preserves them;
- if Y has $\perp$ then Y^X has $\perp$, for any X.

So, also $\mathbb{CPO}_\perp$ is cartesian closed and supports therefore interpretation of simply typed λ-calculus (see chapter 7) and recursion (by the fixed point property). However, the structure of cpo's is much richer than that. First, we shall see that by restricting the morphisms of $\mathbb{CPO}_\perp$ we get a "cpo of cpo's". This will then, later, allow us to solve recursive domain equations like:

$$\begin{array}{rl} X \cong 1 + A \times X & \text{lists on alphabet } A \\ X \cong X^X & \text{untyped } \lambda\text{-calculus} \end{array}$$

First we have to go through some technique.

Definition 8.6 *Let P and Q posets. A pair* $P \underset{i}{\overset{r}{\leftrightarrows}} Q$ *of monotone maps is called an* embedding-projection pair *(e-p pair for short), where i is the embedding, r the projection, if $i \dashv r$ and i is full and faithful; equivalently: $ri = \mathrm{id}_P$ and $ir \leq \mathrm{id}_Q$.*

By uniqueness of adjoints, each member of an e-p pair determines the other. It is evident that $\langle \mathrm{id}_P, \mathrm{id}_P\rangle$ is an e-p pair, and that e-p pairs compose. We can therefore define a category $\mathbb{CPO}_\perp^{\mathrm{EP}}$: objects are cpo's with $\perp$, and morphisms $P \to Q$ are e-p pairs $P \underset{i}{\overset{r}{\leftrightarrows}} Q$ such that both i and r are continuous.

Lemma 8.7 *Let* $P \underset{i}{\overset{r}{\leftrightarrows}} Q$ *be an e-p pair, where P and Q are cpo's with $\perp$. Then both i and r are strict, and i is continuous.*

Proof. Being a left adjoint, i preserves colimits, so i is strict and continuous; since $ri = \mathrm{id}_P$ we also have $r(\bot_Q) = ri(\bot_P) = \bot_P$. ∎

For the following theorem, recall that every diagram in $\mathbb{CPO}_\bot$ with *strict* continuous maps will have a limit in $\mathbb{CPO}_\bot$, since this takes place in $(\mathbb{CPO}_\bot)_s$ which is monadic over $\mathbb{CPO}$.

Theorem 8.8 ($\mathbb{CPO}_\bot^{\mathrm{EP}}$ as "cpo of cpo's"; limit-colimit coincidence)

a) *Any chain of maps*

$$P_1 \underset{i_1}{\overset{r_1}{\rightleftarrows}} P_2 \underset{i_2}{\overset{r_2}{\rightleftarrows}} P_3 \underset{i_3}{\overset{r_3}{\rightleftarrows}} \cdots$$

in $\mathbb{CPO}_\bot^{\mathrm{EP}}$ has a colimit in $\mathbb{CPO}_\bot^{\mathrm{EP}}$;

b) *the vertex of this colimit, P, is the limit (in $\mathbb{CPO}_\bot$) of the diagram*

$$P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} cdots$$

and P is also the colimit in $\mathbb{CPO}_\bot$ of the diagram

$$P_1 \xrightarrow{i_1} P_2 \xrightarrow{i_2} \cdots$$

Proof. We prove a) and b) simultaneously. Note that the limit of $P_1 \overset{r_1}{\leftarrow} P_2 \overset{r_2}{\leftarrow} \ldots$ exists in $\mathbb{CPO}_\bot$ since all maps are strict by lemma 8.7; it is the object $P = \{(p_1, p_2, \ldots) \in \prod_{n\geq 1} P_n \,|\, \forall i \geq 1\, r_i(p_{i+1}) = p_i\}$ with pointwise order.

For any k we have maps $P_k \underset{e_k}{\overset{\pi_k}{\rightleftarrows}} P$ where π_k is the k-th projection, and e_k is defined by:

$$(e_k(p))_j = \begin{cases} r_j r_{j+1} \cdots r_{k_1}(p) & \text{if } j < k \\ p & \text{if } j = k \\ i_{j_1} \cdots i_k(p) & \text{if } j > k \end{cases}$$

Now $\pi_k e_k(p) = (e_k(p))_k = p$ and

$$\begin{array}{rcl} e_k\pi_k(p_1, p_2, \ldots) & = & (p_1, p_2, \ldots, p_k, i_k r_k(p_{k+1}), i_{k+1} i_k r_k r_{k+1}(p_{k+2}), \ldots) \\ & \leq & (p_1, p_2, \ldots) \end{array}$$

So $\langle e_k, \pi_k\rangle$ is an e-p pair. Since obviously $r_i\pi_{i+1} = \pi_i$ for all $i \in \mathbb{N}$, also $e_{k+1} i_k = e_k$ must hold (since one component of an e-p pair uniquely determines the other), hence

$$\{P_k \overset{\langle e_k, \pi_k\rangle}{\rightarrow} P \,|\, k \geq 1\}$$

is a cocone in $\mathbb{CPO}_\bot^{\mathrm{EP}}$ for the given chain.

Suppose now that $\{P_k \overset{\langle j_k, s_k\rangle}{\rightarrow} Q \,|\, k \geq 1\}$ is another cocone for the chain in $\mathbb{CPO}_\bot^{\mathrm{EP}}$. Immediately, we have (since P is the limit of $P_1 \overset{r_1}{\leftarrow} P_2 \overset{r_2}{\leftarrow} \cdots$) a unique

$\sigma : Q \to P$ such that $s_k = \pi_k \sigma$ for all k; $\sigma(q) = (s_1(q), s_2(q), \ldots)$. Note that σ is continuous. Since we have a cocone, for any $(p_1, p_2, \ldots)$ in P we have that in Q:

$$j_k(p_k) = j_{k+1} i_k(p_k) = j_{k+1} i_k r_k(p_{k+1}) \leq j_{k+1}(p_{k+1})$$

so $j_1(p_1) \leq j_2(p_2) \leq \ldots$ and we define $J : P \to Q$ by

$$J(p_1, p_2, \ldots) = \bigsqcup_k j_k(p_k)$$

Then $J(\sigma(q)) = \bigsqcup_k j_k s_k(q) \leq q$ because $\langle j_k, s_k \rangle$ is an e-p pair, and by continuity of s_n,

$$\begin{array}{rcl} (\sigma J(p_1, p_2, \ldots))_n & = & s_n(\bigsqcup_k j_k(p_k)) \quad = \\ s_n(\bigsqcup_{k \geq n} j_k(p_k)) & = & \bigsqcup_{k \geq n} s_n j_k(p_k) \end{array}$$

For $k \geq n$ write $\langle i_{nk}, r_{nk} \rangle$ for the e-p pair $P_n \longleftrightarrow P_k$. Using that $r_{nk} s_k = s_n$, $s_k j_k = \mathrm{id}_{P_k}$,

$$\begin{array}{rcl} \bigsqcup_{k \geq n} s_n j_k(p_k) & = & \bigsqcup_{k \geq n} r_{nk} s_k j_k(p_k) \quad = \\ \bigsqcup_{k \geq n} r_{nk}(p_k) & = & \bigsqcup_{k \geq n} p_n \quad = p_n \end{array}$$

So $\sigma J = \mathrm{id}_P$; i.e. the cocone with vertex Q factors uniquely through the one with vertex P; hence the latter is colimiting.

The only thing which remains to be proven, is that $\{P_k \stackrel{e_k}{\to} P \mid k \geq 1\}$ is also a colimiting cocone in $\mathbb{CPO}_\perp$. Firstly, from the definition of $P_k \underset{e_k}{\overset{\pi_k}{\longleftrightarrow}} P$ and the already seen

$$e_k \pi_k(p_1, p_2, \ldots) = (p_1, \ldots, p_k, i_k r_k(p_{k+1}), i_{k+1} i_k r_k r_{k+1}(p_{k+2}), \ldots)$$

it is immediate that, in P,

$$(p_1, p_2, \ldots) = \bigsqcup_{k \geq 1} e_k \pi_k(p_1, p_2, \ldots) = \bigsqcup_{k \geq 1} e_k(p_k)$$

So if $\{P_k \stackrel{f_k}{\to} Q \mid k \geq 1\}$ is another cocone in $\mathbb{CPO}_\perp$ we can define a continuous factorization $P \stackrel{f}{\to} Q$ by

$$f(p_1, p_2, \ldots) = \bigsqcup_k f_k(p_k)$$

but in fact we have no other choice, hence the factorization is unique. ∎

Define an ω-*category* as a category where every chain of maps

$$A_1 \stackrel{f_1}{\to} A_2 \stackrel{f_2}{\to} A_3 \to \cdots$$

has a colimiting cocone; and call a functor between ω-categories *continuous* if it preserves colimits of chains. Theorem 8.8 says that $\mathbb{CPO}_\perp^{\mathrm{EP}}$ is an ω-category.

Lemma 8.9 *Let $\mathcal{A}$ be an ω-category and $F : \mathcal{A} \to \mathcal{A}$ continuous. If $A \in \mathcal{A}_0$ and $A \xrightarrow{f} F(A)$ a map, and the chain*

$$A \xrightarrow{f} F(A) \xrightarrow{F(f)} F^2(A) \xrightarrow{F^2(f)} F^3(A) \to \ldots$$

has colimit with vertex D, then D is isomorphic to $F(D)$.

In particular, if $\mathcal{A}$ has an initial object, F has an up to isomorphism fixed point.

Exercise 141 Prove lemma 8.9. It generalizes the idea of the fixed point property for cpo's.

For any endofunctor $F : \mathcal{A} \to \mathcal{A}$ we define the category F-Alg of F-algebras in a similar way as for a monad, but simpler (since the functor has less structure than the monad): objects are maps $FX \xrightarrow{h} X$, just like that, and maps $(X, h) \to (Y, k)$ are maps $f : X \to Y$ in $\mathcal{A}$ such that

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "h"'] & F(Y) \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. We have:

Lemma 8.10 (Lambek's Lemma) *If $F(X) \xrightarrow{h} X$ is an initial object of F-Alg, then h is an isomorphism in $\mathcal{A}$.*

Proof. $F^2(X) \xrightarrow{F(h)} F(X)$ is also an F-algebra, so there is a unique $k : X \to F(X)$ such that

$$\begin{tikzcd} F(X) \arrow[r, "F(k)"] \arrow[d, "h"'] & F^2(X) \arrow[d, "F(h)"] \\ X \arrow[r, "k"'] & F(X) \end{tikzcd}$$

commutes. Since also

$$\begin{tikzcd} F^2(X) \arrow[r, "F(h)"] \arrow[d, "F(h)"'] & F(X) \arrow[d, "h"] \\ F(X) \arrow[r, "h"'] & X \end{tikzcd}$$

commutes, h is a map of F-algebras: $(F(X), F(h)) \to (X, h)$. Therefore is $hk : (X, h) \to (X, h)$ a map in F-Alg and since (X, h) is initial, $hk = \mathrm{id}_X$. Then $kh = F(h)F(k) = F(hk) = F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$, so h is iso with inverse k. ∎

Exercise 142 In the situation of lemma 8.9, i.e. $\mathcal{A}$ an ω-category, $F : \mathcal{A} \to \mathcal{A}$ continuous, the colimit of

$$0 \xrightarrow{!} F(0) \xrightarrow{F(!)} F^2(0) \xrightarrow{F^2(!)} F^3(0) \to \ldots$$

(where 0 is initial in $\mathcal{A}$, and ! the unique map $0 \to F(0)$) gives the initial F-algebra.

In view of Lambek's Lemma and other considerations (such as the desirability of induction priciples for elements of recursively defined domains), we aim to solve an equation:

$$X \cong F(X)$$

as an initial F-algebra. So we have seen, that as long as F is a continuous functor, we do have initial F-algebras in $\mathbb{CPO}_\perp^{\mathrm{EP}}$. But this in itself did not require the introduction of $\mathbb{CPO}_\perp^{\mathrm{EP}}$ for also $\mathbb{CPO}$ is an ω-category with an initial object (as is, by the way, Set). The force of the embedding-projection pairs resides in the possibilities of handling "mixed variance". Since the expression X^X does not define a functor but in general, the expression X^Y defines a functor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, one says that in X^X, the variable X occurs both covariantly and contravariantly (or, positively and negatively). We shall see that functors of mixed variance on $\mathbb{CPO}_\perp$, that is: functors $(\mathbb{CPO}_\perp^{\mathrm{op}})^n \times (\mathbb{CPO}_\perp)^m \to \mathbb{CPO}_\perp$, can, under certain conditions, be transformed into continuous covariant functors: $(\mathbb{CPO}_\perp^{\mathrm{EP}})^{n+m} \to \mathbb{CPO}_\perp^{\mathrm{EP}}$. Composition with the diagonal functor $\Delta : \mathbb{CPO}_\perp^{\mathrm{EP}} \to (\mathbb{CPO}_\perp^{\mathrm{EP}})^{n+m}$ gives a continuous endofunctor on $\mathbb{CPO}_\perp^{\mathrm{EP}}$ which has a fixed point (up to isomorphism).

The first ingredient we need is the notion of *local continuity*. Recall that in the proof that $\mathbb{CPO}$ was a ccc, we have basically seen that for cpo's P and Q the set $\mathbb{CPO}(P, Q)$ is itself a cpo. Of course, this holds for $\mathbb{CPO}^{\mathrm{op}}$ too, and also for products of copies of $\mathbb{CPO}$ and $\mathbb{CPO}^{\mathrm{op}}$:

$$(\mathbb{CPO}^{\mathrm{op}})^n \times (\mathbb{CPO})^m((A_1', \ldots, A_n', B_1, \ldots, B_m), (A_1, \ldots, A_n, B_1', \ldots, B_m'))$$

is the cpo $\mathbb{CPO}(A_1, A_1') \times \cdots \times \mathbb{CPO}(B_m, B_m')$.

Definition 8.11 *A functor* $F : (\mathbb{CPO}^{\mathrm{op}})^n \times (\mathbb{CPO})^m \to \mathbb{CPO}$ *is called* locally continuous *if its action on maps:*

$$F_1 : (\mathbb{CPO}^{\mathrm{op}})^n \times (\mathbb{CPO})^m((\vec{A'}, \vec{B}), (\vec{A}, \vec{B'})) \to \mathbb{CPO}(F(\vec{A'}, \vec{B}), F(\vec{A}, \vec{B'}))$$

is a map of cpo's, that is: continuous. We have the same notion if we replace $\mathbb{CPO}$ *by* $\mathbb{CPO}_\perp$.

Example. The product and coproduct functors: $\mathbb{CPO} \times \mathbb{CPO} \to \mathbb{CPO}$, and the exponent functor: $\mathbb{CPO}^{\mathrm{op}} \times \mathbb{CPO} \to \mathbb{CPO}$ are locally continuous.

Theorem 8.12 *Suppose* $F : \mathbb{CPO}_\perp^{\mathrm{op}} \times \mathbb{CPO}_\perp \to \mathbb{CPO}_\perp$ *is locally continuous. Then there is an* $\hat{F} : \mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}} \to \mathbb{CPO}_\perp^{\mathrm{EP}}$ *which is continuous and has the same action on objects as* F.

Proof. We put $\hat{F}(P,Q) = F(P,Q)$, so the last statement of the theorem has been taken care of. Given e-p pairs $P \underset{i}{\overset{r}{\leftrightarrows}} P'$ and $Q \underset{j}{\overset{s}{\leftrightarrows}} Q'$ we have an e-p pair $F(P,Q) \underset{F(r,j)}{\overset{F(i,s)}{\leftrightarrows}} F(P',Q')$ since $F(i,s) \circ F(r,j) = F(r \circ i, s \circ j)$ (recall that F is contravariant in its first argument!) $= F(\mathrm{id}, \mathrm{id}) = \mathrm{id}_{F(P,Q)}$, and $F(r,j) \circ F(i,s) = F(i \circ r, j \circ s) \leq F(\mathrm{id}, \mathrm{id}) = \mathrm{id}_{F(P',Q')}$ (the last inequality is by local continuity of F). So we let

$$\hat{F}(\langle i,r \rangle, \langle j,s \rangle) = \langle F(r,j), F(i,s) \rangle$$

Then clearly, $\hat{F}$ is a functor. To see that $\hat{F}$ is continuous, suppose we have a chain of maps in $\mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}}$: $(A_1, B_1) \to (A_2, B_2) \to \ldots$ with colimit (D,E). That means in $\mathbb{CPO}_\perp^{\mathrm{EP}}$ we have two chains, each with its colimit:

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[dr, "\langle i^1, r^1 \rangle"'] & A_2 \arrow[r] \arrow[d, "\langle i^2, r^2 \rangle \cdots"] & \cdots \\
& D &
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r] \arrow[dr, "\langle j^1, s^1 \rangle"'] & B_2 \arrow[r] \arrow[d, "\langle j^2, s^2 \rangle \cdots"] & \cdots \\
& E &
\end{tikzcd}$$

From the proof of theorem 8.8 we know that $\mathrm{id}_D = \bigsqcup_n i^n \circ r^n$ and $\mathrm{id}_E = \bigsqcup_n j^n \circ s^n$ so $\mathrm{id}_(D,E) = \bigsqcup (i^n \circ r^n, j^n \circ s^n)$. From local continuity of F then, $\mathrm{id}_{\hat{F}(D,E)} = \hat{F}(\mathrm{id}_{(D,E)}) = \langle F(\mathrm{id}_D, \mathrm{id}_E), F(\mathrm{id}_D, \mathrm{id}_E \rangle = \bigsqcup_n \langle F(i^n \circ r^n, j^n \circ s^n), F(i^n \circ r^n, j^n \circ s^n) \rangle$. But this characterizes the colimit of a chain in $\mathbb{CPO}_\perp^{\mathrm{EP}}$, so $\hat{F}(D,E) = F(D,E)$ is isomorphic to the vertex of the colimit in $\mathbb{CPO}_\perp^{\mathrm{EP}}$ of the chain:

$$\hat{F}(A_1, B_1) \to \hat{F}(A_2, B_2) \to \cdots$$

■

Example: a model of the untyped λ-calculus. In the *untyped* λ-calculus, we have a similar formalism as the one given in Chapter 7, but now there are no types. That means, that variables denote functions and arguments at the same time!

In order to model this, we seek a nontrivial solution to:

$$X \cong X^X$$

(Since $X = 1$ is always a solution, "nontrivial" means: not this one) According to theorem 8.12, the exponential functor $\mathbb{CPO}_\perp^{\mathrm{op}} \times \mathbb{CPO}_\perp \to \mathbb{CPO}_\perp$, which sends (X,Y) to Y^X and is locally continuous, gives rise to a continuous functor $\mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}} \overset{\mathrm{Exp}}{\to} \mathbb{CPO}_\perp^{\mathrm{EP}}$. Let $F : \mathbb{CPO}_\perp^{\mathrm{EP}} \to \mathbb{CPO}_\perp^{\mathrm{EP}}$ be the composite

$$\mathbb{CPO}_\perp^{\mathrm{EP}} \overset{\Delta}{\to} \mathbb{CPO}_\perp^{\mathrm{EP}} \times \mathbb{CPO}_\perp^{\mathrm{EP}} \overset{\mathrm{Exp}}{\to} \mathbb{CPO}_\perp^{\mathrm{EP}}$$

Δ (the diagonal functor) is continuous, so F is, since continuous functors compose.

Exercise 143 Show that the functor F works as follows: for an e-p pair $P \underset{i}{\overset{r}{\leftrightarrows}} Q$, $F(\langle i,r\rangle)$ is the e-p pair $P^P \underset{I}{\overset{R}{\leftrightarrows}} Q^Q$ where $I(f) = i \circ f \circ r$ and $R(g) = r \circ g \circ i$.

Let us try and apply lemma 8.9; we need a non-initial object P and a map $P \xrightarrow{f} F(P)$ in $\mathbb{CPO}_{\perp}^{\mathrm{EP}}$.

That is, an embedding-projection pair $P \leftrightarrows P^P$. Well, it is readily checked that for *any* cpo P with $\perp$ the pair $\langle \iota, \rho\rangle$ is an e-p pair, where $\iota : P \to P^P$ sends p to the function which has constant value p, and $\rho : P^P \to P$ sends the function f to $f(\perp)$. We now summarize all our preparations into the final theorem.

Theorem 8.13 (Scott) *Let P be a cpo with $\perp$. Define a diagram*

$$D_0 \underset{i_0}{\overset{r_0}{\leftrightarrows}} D_1 \underset{i_1}{\overset{r_1}{\leftrightarrows}} D_2 \underset{i_2}{\overset{r_2}{\leftrightarrows}} \cdots$$

in $\mathbb{CPO}$ as follows:

- $D_0 = P$; $D_{n+1} = D_n^{D_n}$;
- $i_0 : P \to P^P$ *is* $\lambda p.\lambda q.p$;
- $r_0 : P^P \to P$ *is* $\lambda f.f(\perp)$;
- $i_{n+1} : D_n^{D_n} = D_{n+1} \to D_{n+1}^{D_{n+1}}$ *is* $\lambda f.i_n \circ f \circ r_n$;
- $r_{n+1} : D_{n+1}^{D_{n+1}} \to D_n^{D_n}$ *is* $\lambda g.r_n \circ g \circ i_n$.

Let D_∞ be (vertex of) the limit in $\mathbb{CPO}$ of

$$D_0 \overset{r_0}{\leftarrow} D_1 \overset{r_1}{\leftarrow} D_2 \overset{r_2}{\leftarrow} \cdots$$

Then $D_\infty \cong D_\infty^{D_\infty}$ in $\mathbb{CPO}$.

9 Presheaves

We start by reviewing the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors from $\mathcal{C}$ to Set. $\mathcal{C}$ is assumed to be a small category throughout. Objects of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ are called *presheaves on* $\mathcal{C}$.

We have the Yoneda embedding $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$; we write its effect on objects C and arrows f as y_C, y_f respectively. So for $f : C \to D$ we have $y_f : y_C \to y_D$. Recall: $y_C(C') = \mathcal{C}(C', C)$, the set of arrows $C' \to C$ in $\mathcal{C}$; for $\alpha : C'' \to C'$ we have $y_C(\alpha) : y_C(C') \to y_C(C'')$ which is defined by composition with α, so $y_C(\alpha)(g) = g\alpha$ for $g : C' \to C$. For $f : C \to D$ we have $y_f : y_C \to y_D$ which is a natural transformation with components

$$(y_f)_{C'} : y_C(C') \to y_D(C')$$

given by $(y_f)_{C'}(g) = fg$. Note, that the naturality of y_f is just the associativity of composition in $\mathcal{C}$.

Presheaves of the form y_C are called *representable*.

The *Yoneda Lemma* (2.2) says that there is a 1-1 correspondence between elements of $X(C)$ and arrows in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ from y_C to X, for presheaves X and objects C of $\mathcal{C}$, and this correspondence is natural in both X and C. To every element $x \in X(C)$ corresponds a natural transformation $\mu : y_C \to X$ such that $(\mu)_C(\mathrm{id}_C) = x$; and natural transformations from y_C are completely determined by their effect on id_C. An important consequence of the Yoneda lemma is that the Yoneda embedding is actually an embedding, that is: full and faithful, and injective on objects.

Examples of presheaf categories

1. A first example is the category of presheaves on a monoid (a one-object category) M. Such a presheaf is nothing but a set X together with a right M-*action*, that is: we have a map $X \times M \to X$, written $x, f \mapsto xf$, satisfying $xe = x$ (for the unit e of the monoid), and $(xf)g = x(fg)$. There is only one representable presheaf.

2. If the category C is a poset $(P, \leq)$, for $p \in P$ we have the representable y_p with $y_p(q) = \{*\}$ if $q \leq p$, and $\emptyset$ otherwise. So we can identify the representable y_p with the *downset* $\downarrow(p) = \{q \mid q \leq p\}$.

3. The category of directed graphs and graph morphisms is a presheaf category: it is the category of presheaves on the category with two objects e and v, and two non-identity arrows $\sigma, \tau : v \to e$. For a presheaf X on this category, $X(v)$ can be seen as the set of vertices, $X(e)$ the set of edges, and $X(\sigma), X(\tau) : X(e) \to X(v)$ as the source and target maps.

4. A *tree* is a partially ordered set T with a least element, such that for any $x \in T$, the set $\downarrow(x) = \{y \in T \mid y \leq x\}$ is a finite linearly ordered subset of T. A *morphism* of trees $f : T \to S$ is an order-preserving function wth the property that for any element $x \in T$, the restriction of f to $\downarrow(x)$ is a

bijection from $\downarrow(x)$ to $\downarrow(f(x))$. A *forest* is a set of trees; a map of forests $X \to Y$ is a function $\phi : X \to Y$ together with an X-indexed collection $(f_x \,|\, x \in X)$ of morphisms of trees such that $f_x : x \to \phi(x)$. The category of forests and their maps is just the category of presheaves on ω, the first infinite ordinal.

Recall the definition of the category $y{\downarrow}X$ (an example of a 'comma category' construction): objects are pairs (C, μ) with C an object of $\mathcal{C}$ and $\mu : y_C \to X$ an arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. A morphism $(C, \mu) \to (C', \nu)$ is an arrow $f : C \to C'$ in $\mathcal{C}$ such that the triangle

$$\begin{tikzcd} y_C \arrow[rr, "y_f"] \arrow[dr, "\mu"'] & & y_{C'} \arrow[dl, "\nu"] \\ & X & \end{tikzcd}$$

commutes.

Note that if this is the case and $\mu : y_C \to X$ corresponds to $\xi \in X(C)$ and $\nu : y_{C'} \to X$ corresponds to $\eta \in X(C')$, then $\xi = X(f)(\eta)$.

There is a functor $U_X : y{\downarrow}X \to \mathcal{C}$ (the forgetful functor) which sends (C, μ) to C and f to itself; by composition with y we get a diagram

$$y{\circ}U_X : y{\downarrow}X \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

Clearly, there is a natural transformation ρ from $y{\circ}U_X$ to the constant functor Δ_X from $y{\downarrow}X$ to $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ with value X: let $\rho_{(C,\mu)} = \mu : y_C \to X$. So there is a cocone in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ for $y{\circ}U_X$ with vertex X.

Proposition 9.1 *The cocone $\rho : y{\circ}U_X \Rightarrow \Delta_X$ is colimiting.*

Proof. Suppose $\lambda : y{\circ}U_X \Rightarrow \Delta_Y$ is another cocone. Define $\nu : X \to Y$ by $\nu_C(\xi) = (\lambda_{(C,\mu)})_C(\mathrm{id}_C)$, where $\mu : y_C \to X$ corresponds to ξ in the Yoneda Lemma.

Then ν is natural: if $f : C' \to C$ in $\mathcal{C}$ and $\mu' : y_{C'} \to X$ corresponds to $X(f)(\xi)$, the diagram

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\mu'"'] & & y_C \arrow[dl, "\mu"] \\ & X & \end{tikzcd}$$

commutes, so f is an arrow $(C', \mu') \to (C, \mu)$ in $y{\downarrow}X$. Since λ is a cocone, we have that

$$\begin{tikzcd} y_{C'} \arrow[rr, "y_f"] \arrow[dr, "\lambda_{(C',\mu')}"'] & & y_C \arrow[dl, "\lambda_{(C,\mu)}"] \\ & Y & \end{tikzcd}$$

commutes; so

$$\begin{array}{lcrc}
\nu_{C'}(X(f)(\xi)) & = & (\lambda_{(C',\mu')})_{C'}(\mathrm{id}_{C'}) & = \\
(\lambda_{(C,\mu)})_{C'}((y_f)_{C'}(\mathrm{id}_{C'})) & = & (\lambda_{(C,\mu)})_{C'}(f) & = \\
Y(f)((\lambda_{(C,\mu)})_{C}(\mathrm{id}_{C})) & = & Y(f)(\nu_C(\xi)) &
\end{array}$$

It is easy to see that $\lambda : y \circ U_X \Rightarrow \Delta_Y$ factors through ρ via ν, and that the factorization is unique. ∎

Proposition 9.1 is often referred to by saying that "every presheaf is a colimit of representables".

Let us note that the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is complete and cocomplete, and that limits and colimits are calculated 'pointwise': if I is a small category and $F : I \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a diagram, then for every object C of $\mathcal{C}$ we have a diagram $F_C : I \to Set$ by $F_C(i) = F(i)(C)$; if X_C is a colimit for this diagram in Set, there is a unique presheaf structure on the collection $(X_C \mid C \in \mathcal{C}_0)$ making it into the vertex of a colimit for F. The same holds for limits. Some immediate consequences of this are:

i) an arrow $\mu : X \to Y$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is mono (resp. epi) if and only if every component μ_C is an injective (resp. surjective) function of sets;

ii) the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is regular, and every epimorphism is a regular epi;

iii) the initial object of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the constant presheaf with value $\emptyset$;

iv) X is terminal in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ if and only if every set $X(C)$ is a singleton;

v) for every presheaf X, the functor $(-) \times X : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves colimits.

Furthermore we note the following fact: the Yoneda embedding $\mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is the 'free colimit completion' of $\mathcal{C}$. That is: for any functor $F : \mathcal{C} \to \mathcal{D}$ where $\mathcal{D}$ is a cocomplete category, there is, up to isomorphism, exactly one *colimit preserving* functor $\tilde{F} : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{D}$ such that the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "y"'] & & \mathcal{D} \\
& \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \arrow[ur, "\tilde{F}"'] &
\end{tikzcd}$$

commutes. $\tilde{F}(X)$ is computed as the colimit in $\mathcal{D}$ of the diagram

$$y{\downarrow}X \stackrel{U_X}{\to} \mathcal{C} \stackrel{F}{\to} \mathcal{D}$$

The functor $\tilde{F}$ is also called the 'left Kan extension of F along y'.

We shall now calculate explicitly some structure of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Exponentials can be calculated using the Yoneda Lemma and proposition 9.1. For Y^X, we need a natural 1-1 correspondence

$$\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(Z, Y^X) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(Z \times X, Y)$$

In particular this should hold for representable presheaves y_C; so, by the Yoneda Lemma, we should have a 1-1 correspondence

$$Y^X(C) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$$

which is natural in C. This leads us to *define* a presheaf Y^X by: $Y^X(C) = \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$, and for $f : C' \to C$ we let $Y^X(f) : Y^X(C) \to Y^X(C')$ be defined by composition with $y_f \times \mathrm{id}_X : y_{C'} \times X \to y_C \times X$. Then certainly, Y^X is a well-defined presheaf and for representable presheaves we have the natural bijection $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C, Y^X) \simeq \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}(y_C \times X, Y)$ we want. In order to show that it holds for arbitrary presheaves Z we use proposition 9.1. Given Z, we have the diagram $y \circ U_Z : y{\downarrow}Z \to \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of which Z is a colimit. Therefore arrows $Z \to Y^X$ correspond to cocones on $y \circ U_Z$ with vertex Y^X. Since we have our correspondence for representables y_C, such cocones correspond to cocones on the diagram

$$y{\downarrow}Z \stackrel{U_Z}{\to} \mathcal{C} \stackrel{y}{\to} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \stackrel{(-)\times X}{\to} \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$$

with vertex Y. Because, as already noted, the functor $(-)\times X$ preserves colimits, these correspond to arrows $Z \times X \to Y$, as desired.

It is easy to see that the construction of Y^X gives a functor $(-)^X : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ which is right adjoint to $(-)\times X$, thus establishing that $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is cartesian closed. The *evaluation map* $\mathrm{ev}_{X,Y} : Y^X \times X \to Y$ is given by

$$(\phi, x) \ \mapsto \ \phi_C(\mathrm{id}_C, x)$$

Exercise 144 Show that the map $\mathrm{ev}_{X,Y}$, thus defined, is indeed a natural transformation.

Exercise 145 Prove that $y : \mathcal{C} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ preserves all limits which exist in $\mathcal{C}$. Prove also, that if $\mathcal{C}$ is cartesian closed, y preserves exponents.

Another piece of structure we shall need is that of a *subobject classifier.*

Suppose $\mathcal{E}$ is a category with finite limits. A subobject classifier is a monomorphism $t : T \to \Omega$ with the property that for any monomorphism $m : A \to B$ in $\mathcal{E}$ there is a *unique* arrow $\phi : B \to \Omega$ such that there is a pullback diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "m"'] & T \arrow[d, "t"] \\ B \arrow[r, "\phi"'] & \Omega \end{tikzcd}$$

We say that the unique arrow ϕ *classifies* m or rather, the subobject represented by m (if m and m' represent the same subobject, they have the same classifying arrow). In Set, any two element set $\{a, b\}$ together with a specific choice of one of them, say b (considered as arrow $1 \to \{a, b\}$) acts as a subobject classifier: for $A \subset B$ we have the unique *characteristic function* $\phi_A : B \to \{a, b\}$ defined by $\phi_A(x) = b$ if $x \in A$, and $\phi_A(x) = a$ otherwise.

It is no coincidence that in Set, the domain of $t : T \to \Omega$ is a terminal object: T is always terminal. For, for any object A the arrow $\phi : A \to \Omega$ which classifies the identity on A factors as tn for some $n : A \to T$. On the other hand, if $k : A \to T$ is any arrow, then we have pullback diagrams

$$\begin{tikzcd} A \arrow[r, "k"] \arrow[d, "\mathrm{id}_A"'] & T \arrow[r, "\mathrm{id}_T"] \arrow[d, "\mathrm{id}_T"] & T \arrow[d, "t"] \\ A \arrow[r, "k"'] & T \arrow[r, "t"'] & \Omega \end{tikzcd}$$

so tk classifies id_A. By uniqueness of the classifying map, $tn = tk$; since t is mono, $n = k$. So T is terminal. Henceforth we shall write $1 \stackrel{t}{\to} \Omega$ for the subobject classifier, or, by abuse of language, just Ω.

Note: if $1 \stackrel{t}{\to} \Omega$ is a subobject classifier in $\mathcal{E}$ then we have a 1-1 correspondence between arrows $A \stackrel{\phi}{\to} \Omega$ and *subobjects* of A. This correspondence is natural in the following sense: given $f : B \to A$ and a subobject U of A; by $f^\sharp(U)$ we denote the subobject of B obtained by pulling back U along f. Then if ϕ classifies U, ϕf classifies $f^\sharp(U)$.

First a remark about subobjects in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. A subobject of X can be identified with a *subpresheaf* of X: that is, a presheaf Y such that $Y(C) \subseteq X(C)$ for each C, and $Y(f)$ is the restriction of $X(f)$ to $Y(\mathrm{cod}(f))$. This follows easily from epi-mono factorizations pointwise, and the corresponding fact in Set.

Again, we use the Yoneda Lemma to compute the subobject classifier in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. We need a presheaf Ω such that at least for each representable presheaf y_C, $\Omega(C)$ is in 1-1 correspondence with the set of subobjects (in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$) of y_C. So we define Ω such that $\Omega(C)$ is the set of subpresheaves of y_C; for $f : C' \to C$ we have $\Omega(f)$ defined by the action of pulling back along y_f.

What do subpresheaves of y_C look like? If R is a subpresheaf of y_C then R can be seen as a set of arrows with codomain C such that if $f : C' \to C$ is in R and $g : C'' \to C'$ is arbitrary, then fg is in R (for, $fg = y_C(g)(f)$). Such a set of arrows is called a *sieve* on C.

Under the correspondence between subobjects of y_C and sieves on C, the operation of pulling back a subobject along a map y_f (for $f : C' \to C$) sends a sieve R on C to the sieve $f^*(R)$ on C' defined by

$$f^*(R) \;=\; \{g : D \to C' \,|\, fg \in R\}$$

So Ω can be defined as follows: $\Omega(C)$ is the set of sieves on C, and $\Omega(f)(R) = f^*(R)$. The map $t : 1 \to \Omega$ sends, for each C, the unique element of $1(C)$ to the maximal sieve on C (i.e., the unique sieve which contains id_C).

Exercise 146 Suppose $\mathcal{C}$ is a preorder $(P, \leq)$. For $p \in P$ we let $\downarrow(p) = \{q \in P \,|\, q \leq p\}$. Show that sieves on p can be identified with downwards closed subsets of $\downarrow(p)$. If we denote the unique arrow $q \to p$ by qp and U is a downwards closed subset of $\downarrow(p)$, what is $(qp)^*(U)$?

Let us now prove that $t : 1 \to \Omega$, thus defined, is a subobject classifier in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Let Y be a subpresheaf of X. Then for any C and any $x \in X(C)$, the set

$$\phi_C(x) \;=\; \{f : D \to C \mid X(f)(x) \in Y(D)\}$$

is a sieve on C, and defining $\phi : X \to \Omega$ in this way gives a natural transformation: for $f : C' \to C$ we have

$$\begin{array}{rcr} \phi_{C'}(X(f)(x)) & = & \{g : D \to C' \mid X(g)(X(f)(x)) \in Y(D)\} \\ & = & \{g : D \to C' \mid X(gf)(x) \in Y(D)\} \\ & = & \{g : D \to C' \mid fg \in \phi_C(x)\} \\ & = & f^*(\phi_C(x)) \\ & = & \Omega(f)(\phi_C(x)) \end{array}$$

Moreover, if we take the pullback of t along ϕ, we get the subpresheaf of X consisting of (at each object C) of those elements x for which $\mathrm{id}_C \in \phi_C(x)$; that is, we get Y. So ϕ classifies the subpresheaf Y.

On the other hand, if $\phi : X \to \Omega$ is any natural transformation such that pulling back t along ϕ gives Y, then for every $x \in X(C)$ we have that $x \in Y(C)$ if and only if $\mathrm{id}_C \in \phi_C(x)$. But then by naturality we get for any $f : C' \to C$ that

$$X(f)(x) \in Y(C') \;\Leftrightarrow\; \mathrm{id}_{C'} \in f^*(\phi_C(x)) \;\Leftrightarrow\; f \in \phi_C(x)$$

which shows that the classifying map ϕ is unique.

Combining the subobject classifier with the cartesian closed structure, we obtain *power objects*. In a category $\mathcal{E}$ with finite products, we call an object A a *power object* of the object X, if there is a natural 1-1 correspondence

$$\mathcal{E}(Y, A) \simeq \mathrm{Sub}_{\mathcal{E}}(Y \times X)$$

The naturality means that if $f : Y \to A$ and $g : Z \to Y$ are arrows in $\mathcal{E}$ and f corresponds to the subobject U of $Y \times X$, then $fg : Z \to A$ corresponds to the subobject $(g \times \mathrm{id}_X)^\sharp(U)$ of $Z \times X$.

Power objects are unique up to isomorphism; the power object of X, if it exists, is usually denoted $\mathcal{P}(X)$. Note the following consequence of the definition: to the identity map on $\mathcal{P}(X)$ corresponds a subobject of $\mathcal{P}(X) \times X$ which we call the "element relation" $\in_X$; it has the property that whenever $f : Y \to \mathcal{P}(X)$ corresponds to the subobject U of $Y \times X$, then $U = (f \times \mathrm{id}_X)^\sharp(\in_X)$.

Convince yourself that power objects in the category Set are just the familiar power sets.

In a cartesian closed category with subobject classifier Ω, power objects exist: let $\mathcal{P}(X) = \Omega^X$. Clearly, the defining 1-1 correspondence is there.

$$\mathcal{P}(X)(C) \;=\; \mathrm{Sub}(y_C \times X)$$

with action $\mathcal{P}(X)(f)(U) = (y_f \times \mathrm{id}_X)^\sharp(U)$.

Exercise 147 Show that $\mathcal{P}(X)(C) = \mathrm{Sub}(y_C \times X)$ and that, for $f : C' \to C$, $\mathcal{P}(X)(f)(U) = (y_f \times \mathrm{id}_X)^\sharp(U)$. Prove also, that the element relation, as a subpresheaf $\in_X$ of $\mathcal{P}(X) \times X$, is given by

$$(\in_X)(C) \;=\; \{(U, x) \in \mathrm{Sub}(y_C \times X) \times X(C) \,|\, (\mathrm{id}_C, x) \in U(C)\}$$

Definition 9.2 A *topos* is a category with finite limits, which is cartesian closed and has a subobject classifier.

Exercise 148 Let $\mathcal{C}$ be a small category; we work in the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of presheaves on $\mathcal{C}$. Let P be such a presheaf. We define a presheaf $\tilde{P}$ as follows: for an object C of $\mathcal{C}$, $\tilde{P}(C)$ consists of those subobjects α of $y_C \times P$ which satisfy the following condition: for all arrows $f : D \to C$, the set

$$\{y \in P(D) \,|\, (f, y) \in \alpha(D)\}$$

has at most one element.

a) Complete the definition of $\tilde{P}$ as a presheaf.

b) Show that there is a monic map $\eta_P : P \to \tilde{P}$ with the following property: for every diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "m"'] & P \\ B & \end{tikzcd}$$

with m mono, there is a unique map $\tilde{g} : B \to \tilde{P}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "m"'] & P \arrow[d, "\eta_P"] \\ B \arrow[r, "\tilde{g}"'] & \tilde{P} \end{tikzcd}$$

is a pullback square. The arrow $P \xrightarrow{\eta_P} \tilde{P}$ is called a *partial map classifier* for P.

c) Show that the assignment $P \mapsto \tilde{P}$ is part of a functor $\tilde{(\cdot)}$ in such a way that the maps η_P form a natural transformation from the identity functor to $\tilde{(\cdot)}$, and all naturality squares for η are pullbacks.

Exercise 149 Let $\mathcal{E}$ be a topos with subobject classifier $1 \xrightarrow{t} \Omega$.

a) Prove that Ω is injective.

b) Prove that every object of the form Ω^X is injective.

c) Conclude that $\mathcal{E}$ has enough injectives.

9.1 Recovering the category from its presheaves?

In this short section we shall see to what extent the category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ determines $\mathcal{C}$. In other words, suppose $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ are equivalent categories; what can we say about $\mathcal{C}$ and $\mathcal{D}$?

Definition 9.3 In a regular category an object P is called (regular) *projective* if for every regular epi $f : A \to B$, any arrow $P \to B$ factors through f. Equivalently, every regular epi with codomain P has a section.

Exercise 150 Prove the equivalence claimed in definiton 9.3.

Definition 9.4 An object X is called *indecomposable* if whenever X is a coproduct $\coprod_i U_i$, then for *exactly* one i the object U_i is not initial.

Note, that an initial object is not indecomposable, just as 1 is not a prime number.

In $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, coproducts are *stable*, which means that they are preserved by pullback functors; this is easy to check. Another triviality is that the initial object is *strict*: the only maps into it are isomorphisms.

Proposition 9.5 *In* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$*, a presheaf* X *is indecomposable and projective if and only if it is a retract of a representable presheaf: there is a diagram* $X \xrightarrow{i} y_C \xrightarrow{r} X$ *with* $ri = \mathrm{id}_X$.

Proof. Check yourself that every retract of a projective object is again projective. Similarly, a retract of an indecomposable object is indecomposable: if $X \xrightarrow{i} Y \xrightarrow{r} X$ is such that $ri = \mathrm{id}_X$ and Y is indecomposable, any presentation of X as a coproduct $\coprod_i U_i$ can be pulled back along r to produce, by stability of coproducts, a presentation of Y as coproduct $\coprod_i V_i$ such that

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & Y \arrow[d, "r"] \\ U_i \arrow[r] & X \end{tikzcd}$$

is a pullback; for exactly one i then, V_i is non-initial; hence since r is epi and the initial object is strict, for exactly one i we have that U_i is non-initial. We see that the property of being projective and indecomposable is inherited by retracts. Moreover, every representable is indecomposable and projective, as we leave for you to check.

Conversely, assume X is indecomposable and projective. By proposition 9.1 and the standard construction of colimits from coproducts and coequalizers, there is an epi $\coprod_i y_{C_i} \to X$ from a coproduct of representables. Since X is projective, this epi has a section ι. Pulling back along ι we get a presentation of X as a coproduct $\coprod_i V_i$ such that

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & X \arrow[d, "\iota"] \\ y_{C_i} \arrow[r] & \coprod_i y_{C_i} \end{tikzcd}$$

is a pullback diagram. X was assumed indecomposable, so exactly one V_i is non-initial. But this means that X is a retract of y_{C_i}. ∎

If X is a retract of y_C, say $X \xrightarrow{\mu} y_C \xrightarrow{\nu} X$ with $\nu\mu = \mathrm{id}_X$, consider $\mu\nu : y_C \to y_C$. This arrow is *idempotent*: $(\mu\nu)(\mu\nu) = \mu(\nu\mu)\nu = \mu\nu$, and since the Yoneda embedding is full and faithful, $\mu\nu = y_e$ for an idempotent $e : C \to C$ in $\mathcal{C}$.

A category $\mathcal{C}$ is said to be *Cauchy complete* if for every idempotent $e : C \to C$ there is a diagram $D \xrightarrow{i} C \xrightarrow{r} D$ with $ri = \mathrm{id}_D$ and $ir = e$. One also says: "idempotents split". In the situation above (where X is a retract of y_C) we see that X must then be isomorphic to y_D for a retract D of C in $\mathcal{C}$. We conclude:

Theorem 9.6 *If $\mathcal{C}$ is Cauchy complete, $\mathcal{C}$ is equivalent to the full subcategory of* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *on the indecomposable projectives. Hence if $\mathcal{C}$ and $\mathcal{D}$ are Cauchy complete and* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *and* $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$ *are equivalent, so are $\mathcal{C}$ and $\mathcal{D}$.*

Exercise 151 Show that if $\mathcal{C}$ has equalizers, $\mathcal{C}$ is Cauchy complete.

9.2 The Logic of Presheaves

Definition 9.7 A *Heyting algebra H* is a lattice $(\bot, \top, \vee, \wedge)$ together with a binary operation $\to$ (called *Heyting implication*), which satisfies the following equivalence for all $a, b, c \in H$:

$$a \wedge b \leq c \;\Leftrightarrow\; a \leq b \to c$$

Exercise 152 Prove that every Heyting algebra, as a lattice, is *distributive*: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ hold, for all $x, y, z \in H$.

For a presheaf X we shall write $\mathrm{Sub}(X)$ for the set of subpresheaves of X. So if $\phi : X \to Y$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ we gave $\phi^\sharp : \mathrm{Sub}(Y) \to \mathrm{Sub}(X)$ by pulling back.

Theorem 9.8 *Every poset* $\mathrm{Sub}(X)$ *is a Heyting algebra. For every $\phi : Y \to X$, the map $\phi^\sharp$ commutes with the Heyting structure $(\bot, \top, \wedge, \vee, \to)$. Moreover, $\phi^\sharp$ has both a right and a left adjoint, denoted $\forall_\phi$ and $\exists_\phi$ respectively.*

Proof. Since limits and colimits are computed pointwise, $\wedge$ and $\vee$ (between subpresheaves) are given by pointwise intersection and union, respectively. The empty subpresheaf is, of course, the bottom element of $\mathrm{Sub}(X)$, and X itself is the top element. Heyting implication is *not* done pointwise, since if A and B are subpresheaves of X, setting

$$(A \to B)(C) \ = \ \{x \in X(C) \,|\, x \in A(C) \Rightarrow x \in B(C)\}$$

does not necessarily define a subpresheaf of X (check this!). Therefore we put

$$(A \to B)(C) \ = \ \{x \in X(C) \,|\, \forall f : C' \to C(X(f)(x) \in A(C') \Rightarrow X(f)(x) \in B(C'))\}$$

Then $(A \to B)$ is a subpresheaf of X. It is easy to verify that if D is another subpresheaf of X then D is a subpresheaf of $(A \to B)$ if and only if $D \cap A$ is a subpresheaf of B.

Let us check that $\phi^\sharp$ preserves Heyting implication (the rest of the structure is left to you):

$$\begin{array}{rcl} (\phi^\sharp(A) \to \phi^\sharp(B))(C) & = & \begin{array}{r}\{y \in Y(C)\,\forall f : C' \to C(Y(f)(y) \in \phi^\sharp(A)(C') \Rightarrow \\ Y(f)(y) \in \phi^\sharp(B)(C'))\}\end{array} \\ & = & \begin{array}{r}\{y \in Y(C) \,|\, \forall f : C' \to C(\phi_{C'}(Y(f)(y)) \in A(C') \Rightarrow \\ \phi_{C'}(Y(f)(y)) \in B(C'))\}\end{array} \\ & = & \begin{array}{r}\{y \in Y(C) \,|\, \forall f : C' \to C(X(f)(\phi_C(y)) \in A(C') \Rightarrow \\ X(f)(\phi_C(y)) \in B(C'))\}\end{array} \\ & = & \{y \in Y(C) \,|\, \phi_C(y) \in (A \to B)(C)\} \\ & = & \phi^\sharp(A \to B)(C) \end{array}$$

The left adjoint $\exists_\phi(A)$ (where now A is a subpresheaf of Y) is, just as in the case of regular categories, given by the *image* of A under ϕ, and this is done pointwise. So,

$$\exists_\phi(A)(C) \ = \ \{x \in X(C) \,|\, \exists y \in A(C)(x = \phi_C(y))\}$$

Clearly then, if B is a subpresheaf of X, we have $\exists_\phi(A) \leq B$ in $\mathrm{Sub}(X)$ if and only if $A \leq \phi^\sharp(B)$ in $\mathrm{Sub}(Y)$.

The right adjoint $\forall_\phi(A)$ is given by

$$\forall_\phi(A)(C) \ = \ \{x \in X(C) \,|\, \forall f : D \to C \forall y \in Y(D)(\phi_D(y) = x \Rightarrow y \in A(D))\}$$

Check for yourself that then, $B \leq \forall_\phi(A)$ in $\mathrm{Sub}(X)$ if and only if $\phi^\sharp(B) \leq A$ in $\mathrm{Sub}(Y)$. ∎

Exercise 153 Prove that for $\phi : Y \to X$, A a subpresheaf of X and B a subpresheaf of Y,

$$\exists_\phi(\phi^\sharp(A) \wedge B) \ = \ A \wedge \exists_\phi(B)$$

Which property of the map $\phi^\sharp$ do you need?

Exercise 154 Suppose X is a presheaf on $\mathcal{C}$. Let $\mathcal{S}$ be the set of all those $\mathcal{C}_0$-indexed collections of sets $A = (A_C \,|\, c \in \mathcal{C}_0)$ for which A_C is a subset of $X(C)$ for each C. $\mathcal{S}$ is ordered pointwise: $A \leq B$ iff for each C, $A_C \subseteq B_C$.

Let $\iota : \mathrm{Sub}(X) \to \mathcal{S}$ be the inclusion. Show that ι has both a left and a right adjoint.

9.2.1 First-order structures in categories of presheaves

In any regular category which satisfies the properties of theorem 9.8 (such a category is often called a 'Heyting category'), one can extend the interpretation of 'regular logic' in regular categories to full first-order logic. We shall retain as much as possible the notation from chapter 4.

We have a language $\mathcal{L}$, which consists of a collection of *sorts* $S, T, \ldots$, possibly *constants* c^S of sort S, *function symbols* $f : S_1, \ldots, S_n \to S$, and *relation symbols* $R \subseteq S_1, \ldots, S_n$. The definition of formula is extended with the clauses:

i) If φ and ψ are formulas then $(\varphi \vee \psi)$, $(\varphi \to \psi)$ and $\neg\varphi$ are formulas;

ii) if φ is a formula and x^S a variable of sort S then $\forall x^S \varphi$ is a formula.

For the notations $FV(t)$ and $FV(\varphi)$ we refer to the mentioned chapter 4. Again, an interpretation assigns objects $[\![S]\!]$ to the sorts S, arrows to the function symbols and subobjects to relation symbols. This then leads to the definition of the interpretation of a formula φ as a subobject $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$, which is a chosen product of the interpretations of all the sorts of the free variables of φ: if $FV(\varphi) = \{x_1^{S_1}, \ldots, x_n^{S_n}\}$ then $[\![FV(\varphi)]\!] = [\![S_1]\!] \times \cdots \times [\![S_n]\!]$.

The definition of $[\![\varphi]\!]$ of the mentioned chapter 4 is now extended by the clauses:

i) If $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ and $[\![\psi]\!] \to [\![FV(\psi)]\!]$ are given and

$$\begin{tikzcd} {[\![FV(\varphi \wedge \psi)]\!]} \arrow[r, "\pi_1"] \arrow[dr, "\pi_2"'] & {[\![FV(\varphi)]\!]} \\ & {[\![FV(\psi)]\!]} \end{tikzcd}$$

are the projections, then

$$\begin{array}{rcll} [\![\varphi \vee \psi]\!] & = & (\pi_1)^\sharp([\![\varphi]\!]) \vee (\pi_2)^\sharp([\![\psi]\!]) & \text{in } \mathrm{Sub}([\![FV(\varphi \wedge \psi)]\!]) \\ [\![\varphi \to \psi]\!] & = & (\pi_1)^\sharp([\![\varphi]\!]) \to (\pi_2)^\sharp([\![\psi]\!]) & \text{in } \mathrm{Sub}([\![FV(\varphi \wedge \psi)]\!]) \\ [\![\neg\varphi]\!] & = & [\![\varphi]\!] \to \bot & \text{in } \mathrm{Sub}([\![FV(\varphi)]\!]) \end{array}$$

(Note that $FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi \to \psi)$)

ii) if $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ is given and $\pi : [\![FV(\varphi)]\!] \to [\![FV(\exists x\varphi)]\!]$ is the projection, let $FV'(\varphi) = FV(\varphi \wedge x = x)$ and $\pi' : [\![FV'(\varphi)]\!] \to [\![FV(\varphi)]\!]$ the projection. Then

$$[\![\forall x\varphi]\!] \;=\; \forall_{\pi\pi'}((\pi')^\sharp([\![\varphi]\!]))$$

We shall now write out what this means, concretely, in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. For a formula φ, we have $[\![\varphi]\!]$ as a subobject of $[\![FV(\varphi)]\!]$, hence we have a classifying map $\{\varphi\} : [\![FV(\varphi)]\!] \to \Omega$ with components $\{\varphi\}_C : [\![FV(\varphi)]\!](C) \to \Omega(C)$; for $(a_1, \ldots, a_n) \in [\![FV(\varphi)]\!](C)$, $\{\varphi\}_C(a_1, \ldots, a_n)$ is a sieve on C.

Definition 9.9 For φ a formula with free variables $x_1, \ldots, x_n$, C an object of $\mathcal{C}$ and $(a_1, \ldots, a_n) \in [\![FV(\varphi)]\!](C)$, the notation $C \Vdash \varphi(a_1, \ldots, a_n)$ means that $\mathrm{id}_C \in \{\varphi\}_C(a_1, \ldots, a_n)$.

The pronunciation of "$\Vdash$" is 'forces'.

Notation. For φ a formula with free variables $x_1^{S_1}, \ldots, x_n^{S_n}$, C an object of $\mathcal{C}$ and $(a_1, \ldots, a_n) \in [\![FV(\varphi)]\!](C)$ as above, so $a_i \in [\![S_i]\!](C)$, if $f : C' \to C$ is an arrow in $\mathcal{C}$ we shall write $a_i f$ for $[\![S_i]\!](f)(a_i)$.

Note: with this notation and φ, C, $a_1, \ldots, a_n$, $f : C' \to C$ as above, we have $f \in \{\varphi\}_C(a_1, \ldots, a_n)$ if and only if $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$.

Using the characterization of the Heyting structure of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ given in the proof of theorem 9.8, we can easily write down an inductive definition for the notion $C \Vdash \varphi(a_1, \ldots, a_n)$:

- $C \Vdash (t = s)(a_1, \ldots, a_n)$ if and only if $[\![t]\!]_C(a_1, \ldots, a_n) = [\![s]\!]_C(a_1, \ldots, a_n)$
- $C \Vdash R(t_1, \ldots, t_k)(a_1, \ldots, a_n)$ if and only if
$$([\![t_1]\!]_C(a_1, \ldots, a_n), \ldots, [\![t_k]\!]_C(a_1, \ldots, a_n)) \in [\![R]\!](C)$$
- $C \Vdash (\varphi \wedge \psi)(a_1, \ldots, a_n)$ if and only if
$$C \Vdash \varphi(a_1, \ldots, a_n) \text{ and } C \Vdash \psi(a_1, \ldots, a_n)$$
- $C \Vdash (\varphi \vee \psi)(a_1, \ldots, a_n)$ if and only if
$$C \Vdash \varphi(a_1, \ldots, a_n) \text{ or } C \Vdash \psi(a_1, \ldots, a_n)$$
- $C \Vdash (\varphi \to \psi)(a_1, \ldots, a_n)$ if and only if for every arrow $f : C' \to C$,
$$\text{if } C' \Vdash \varphi(a_1 f, \ldots, a_n f) \text{ then } C' \Vdash \psi(a_1 f, \ldots, a_n f)$$
- $C \Vdash \neg\varphi(a_1, \ldots, a_n)$ if and only if for no arrow $f : C' \to C$, $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$
- $C \Vdash \exists x^S \varphi(a_1, \ldots, a_n)$ if and only if for some $a \in [\![S]\!](C)$, $C \Vdash \varphi(a, a_1, \ldots, a_n)$
- $C \Vdash \forall x^S \varphi(a_1, \ldots, a_n)$ if and only if for every arrow $f : C' \to C$ and every $a \in [\![S]\!](C')$,
$$C' \Vdash \varphi(a, a_1 f, \ldots, a_n f)$$

Exercise 155 Prove: if $C \Vdash \varphi(a_1, \ldots, a_n)$ and $f : C' \to C$ is an arrow, then $C' \Vdash \varphi(a_1 f, \ldots, a_n f)$.

Now let ϕ be a sentence of the language, so $[\![\phi]\!]$ is a subobject of 1 in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Note: a subobject of 1 is 'the same thing' as a collection X of objects of $\mathcal{C}$ such that whenever $C \in X$ and $f : C' \to C$ is arbitrary, then $C' \in X$ also. The following theorem is straightforward.

Theorem 9.10 *For a language $\mathcal{L}$ and interpretation $[\![\cdot]\!]$ of $\mathcal{L}$ in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$*, we have that for every $\mathcal{L}$-sentence ϕ, $[\![\phi]\!] = \{C \in \mathcal{C}_0 \mid C \Vdash \phi\}$. Hence, ϕ is true for the interpretation in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *if and only if for every C, $C \Vdash \phi$.*

If Γ is a set of $\mathcal{L}$-sentences and ϕ an $\mathcal{L}$-sentence, we write $\Gamma \Vdash \phi$ to mean: in every interpretation in a presheaf category such that every sentence of Γ is true, ϕ is true.

We mention without proof:

Theorem 9.11 (Soundness and Completeness) *If Γ is a set of $\mathcal{L}$-sentences and ϕ an $\mathcal{L}$-sentence, we have $\Gamma \Vdash \phi$ if and only if ϕ is provable from Γ in intuitionistic predicate calculus.*

Intuitionistic predicate calculus is what one gets from classical logic by deleting the rule which infers ϕ from a proof that $\neg\phi$ implies absurdity. In a Gentzen calculus, this means that one restricts attention to those sequents $\Gamma \Rightarrow \Delta$ for which Δ consists of at most one formula.

Exercise 156 Let N denote the constant presheaf with value $\mathbb{N}$.

i) Show that there are maps $0 : 1 \to N$ and $S : N \to N$ which make N into a natural numbers object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

ii) Accordingly, there is an interpretation of the language of first-order arithmetic in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, where the unique sort is interpreted by N. Prove, that for this interpretation, a sentence in the language of arithmetic is true if and only if it is true classically in the standard model $\mathbb{N}$.

Exercise 157 Prove that for every object C of $\mathcal{C}$, the set $\Omega(C)$ of sieves on C is a Heyting algebra, and that for every map $f : C' \to C$ in $\mathcal{C}$, $\Omega(f) : \Omega(C) \to \Omega(C')$ preserves the Heyting structure. Write out explicitly the Heyting implication $(R \to S)$ of two sieves.

9.3 Two examples and applications

9.3.1 Kripke semantics

Kripke semantics is a special kind of presheaf semantics: $\mathcal{C}$ is taken to be a poset, and the sorts are interpreted by presheaves X such that for every $q \leq p$ the map $X(q \leq p) : X(p) \to X(q)$ is an inclusion of sets. Let us call these presheaves *Kripke presheaves.*

The soundness and completeness theorem 9.11 already holds for Kripke semantics. This raises the question whether the greater generality of presheaves achieves anything new. In this example, we shall see that general presheaves are richer than Kripke models if one considers *intermediate logics*: logics stronger than intuitionistic logic but weaker than classical logic.

In order to warm up, let us look at Kripke models for *propositional logic.* The propositional variables are interpreted as subobjects of 1 in $\mathrm{Set}^{\mathcal{K}^{\mathrm{op}}}$ (for a poset $(\mathcal{K}, \leq)$); that means, as downwards closed subsets of $\mathcal{K}$ (see te remark just before theorem 9.10). Let, for example, $\mathcal{K}$ be the poset:

$$\begin{tikzcd} & 0 & \\ k \arrow[ur] & & l \arrow[ul] \end{tikzcd}$$

and let $[\![p]\!] = \{k\}$. Then $0 \not\Vdash p$, $0 \not\Vdash \neg p$ (since $k \leq 0$ and $k \Vdash p$) and $0 \not\Vdash \neg\neg p$ (since $l \leq 0$ and $l \Vdash \neg p$). So $p \vee \neg p \vee \neg\neg p$ is not true for this interpretation. Even simpler, if $\mathcal{K} = \{0 \leq 1\}$ and $[\![p]\!] = \{0\}$, then $1 \not\Vdash p \vee \neg p$. However, if $\mathcal{K}$ is a linear order, then $(p \to q) \vee (q \to p)$ is always true on $\mathcal{K}$, since if $\mathcal{K}$ is linear, then so is the poset of its downwards closed subsets. From this one can conclude that if one adds to intuitionistic propositional logic the axiom scheme

$$(\phi \to \psi) \vee (\psi \to \phi)$$

one gets a logic which is strictly between intuitionistic and classical logic.

Exercise 158 Prove that $(p \to q) \vee (q \to p)$ is always true on $\mathcal{K}$ if and only if $\mathcal{K}$ has the property that for every $x \in \mathcal{K}$, the set ${\downarrow}x = \{y \in \mathcal{K} \mid y \leq x\}$ is linearly ordered.

Prove also, that $\neg p \vee \neg\neg p$ is always true on $\mathcal{K}$ if and only if $\mathcal{K}$ has the following property: whenever two elements have an upper bound, they also have a lower bound.

Not only certain properties of posets can be characterized by the propositional logic they satisfy in the sense of exercise 158, also properties of presheaves.

Exercise 159 Let X be a Kripke presheaf on a poset $\mathcal{K}$. Show that the following axiom scheme of predicate logic:

$$\mathrm{D} \qquad \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$$

(where A and B may contain additional variables, but the variable x is not allowed to occur in B) is always true in X, if and only if for every $k' \leq k$ in $\mathcal{K}$, the map $X(k) \to X(k')$ is the identity.

Suppose now one considers the logic D-J, which is intuitionistic logic extended with the axiom schemes $\neg\phi \vee \neg\neg\phi$ and the axiom scheme D from exercise 159. One might expect (in view of exercises 158 and 159) that this logic is complete with respect to constant presheaves on posets $\mathcal{K}$ which have the property that whenever two elements have an upper bound, they also have a lower bound. However, this is not the case!

Proposition 9.12 *Suppose X is a constant presheaf on a poset $\mathcal{K}$ which has the property that whenever two elements have an upper bound, they also have a lower bound. Then the following axiom scheme is always true on X:*

$$\begin{array}{c} \forall x[(R \to (S \vee A(x))) \vee (S \to (R \vee A(x)))] \wedge \neg\forall x A(x) \\ \to \\ [(R \to S) \vee (S \to R)] \end{array}$$

Exercise 160 Prove proposition 9.12.

However, the axiom scheme in proposition 9.12 is not a consequence of the logic D-J, which fact can be shown using presheaves. This was also shown by Ghilardi. We give the relevant statements without proof; the interested reader is referred to *Arch.Math.Logic* **29** (1989), 125–136.

Proposition 9.13 *i) The axiom scheme $\neg\phi \vee \neg\neg\phi$ is true in every interpretation in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *if and only if the category $\mathcal{C}$ has the property that every pair of arrows with common codomain fits into a commutative square.*

ii) Let X be a presheaf on a category $\mathcal{C}$. Suppose X has the property that for all $f : C' \to C$ in $\mathcal{C}$, all $n \geq 0$, all $x_1, \ldots, x_n \in X(C)$ and all $y \in X(C')$ there is $f' : C' \to C$ and $x \in X(C)$ such that $xf = y$ and $x_1 f = x_1 f', \ldots, x_n f = x_n f'$. Then for every interpretation on X the axiom scheme D *of exercise 159 is true.*

iii) There exist a category $\mathcal{C}$ satisfying the property of i), and a presheaf X on $\mathcal{C}$ satisfying the property of ii), and an interpretation on X for which an instance of the axiom scheme of proposition 9.12 is not true.

9.3.2 Failure of the Axiom of Choice

In this example, due to M. Fourman and A. Scedrov (*Manuscr. Math.* **38** (1982), 325–332), we explore a bit the higher-order structure of a presheaf category. Recall that the Axiom of Choice says: if X is a set consisting of nonempty sets, there is a function $F : X \to \bigcup X$ such that $F(x) \in x$ for every $x \in X$. This axiom is not provable in Zermelo-Fraenkel set theory, but it is classically totally unproblematic for *finite* X (induction on the cardinality of X).

We exhibit here a category $\mathcal{C}$, a presheaf Y on $\mathcal{C}$, and a subpresheaf X of the power object $\mathcal{P}(Y)$ such that the following statements are true in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$:

$\forall\alpha\beta \in X(\alpha = \beta)$ ("X has at most one element")

$\forall\alpha \in X \exists xy \in Y(x \neq y \wedge \forall z \in Y(z \in \alpha \leftrightarrow z = x \vee z = y))$ ("every element of X has exactly two elements")

There is no arrow $X \to \bigcup X$ (this is stronger than: X has no choice function).

Consider the category $\mathcal{C}$ with two objects and two non-identity arrows:

$$\begin{tikzcd} D \arrow[loop left, "\beta"] \arrow[r, "\alpha"] & E \end{tikzcd}$$

subject to the equations $\beta^2 = \mathrm{id}_D$ and $\alpha\beta = \alpha$.

We calculate the representables y_D and y_E, and the map $y_\alpha : y_D \to y_E$:

$y_D(E) = \emptyset$	$(y_\alpha)_D(\mathrm{id}_D) = \alpha$
$y_D(D) = \{\mathrm{id}_D, \beta\}$	$(y_\alpha)_D(\beta) = \alpha$
$y_D(\alpha)$ is the empty function	$(y_\alpha)_E$ is the empty function
$y_D(\beta)(\mathrm{id}_D) = \beta$	$y_D(\beta)(\beta) = \mathrm{id}_D$

Since E is terminal in $\mathcal{C}$, y_E is a terminal object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$:

$$y_E(E) = \{\mathrm{id}_E\},\ y_E(D) = \{\alpha\},\ y_E(\alpha)(\mathrm{id}_E) = \alpha,\ y_E(\beta)(\alpha) = \alpha$$

Now let us calculate the power object $\mathcal{P}(y_D)$. According to the explicit construction of power objects in presheaf categories, we have

$$\begin{array}{rcl} \mathcal{P}(y_D)(E) & = & \mathrm{Sub}(y_E \times y_D) \\ \mathcal{P}(y_D)(D) & = & \mathrm{Sub}(y_D \times y_D) \end{array}$$

$(y_E \times y_D)(D)$ is the two-element set $\{(\alpha, \mathrm{id}_D), (\alpha, \beta)\}$ which are permuted by the action of β, and $(y_E \times y_D)(E) = \emptyset$. So we see that $\mathrm{Sub}(y_E \times y_D)$ has two elements: $\emptyset$ (the empty presheaf) and $y_E \times y_D$ itself. $(y_D \times y_D)(D)$ has 4 elements: $(\mathrm{id}_D, \beta), (\beta, \mathrm{id}_D), (\beta, \beta), (\mathrm{id}_D, \mathrm{id}_D)$ and we have: $(\mathrm{id}_D, \beta)\beta = (\beta, \mathrm{id}_D)$ and $(\beta, \beta)\beta = (\mathrm{id}_D, \mathrm{id}_D)$.

So $\mathrm{Sub}(y_D \times y_D)$ has 4 elements: $\emptyset, y_D \times y_D, A, B$ where A and B are such that

$$\begin{array}{ll} A(E) = \emptyset & A(D) = \{(\mathrm{id}_D, \beta), (\beta, \mathrm{id}_D)\} \\ B(E) = \emptyset & B(D) = \{(\beta, \beta), (\mathrm{id}_D, \mathrm{id}_D)\} \end{array}$$

Summarizing: we have $\mathcal{P}(y_D)(E) = \{\emptyset, y_E \times y_D\}$, $\mathcal{P}(y_D)(D) = \{\emptyset, y_D \times y_D, A, B\}$. The map $\mathcal{P}(y_D)(\alpha)$ is given by pullback along $y_\alpha \times \mathrm{id}_{y_D}$ and sends therefore $\emptyset$ to $\emptyset$ and $y_E \times y_D$ to $y_D \times y_D$. $\mathcal{P}(y_D)(\beta)$ is by pullback along $y_\beta \times \mathrm{id}_{y_D}$ and sends $\emptyset$ to $\emptyset$, $y_D \times y_D$ to $y_D \times y_D$, and permutes A and B.

Now let X be the subpresheaf of $\mathcal{P}(y_D)$ given by:

$$X(E) = \emptyset \quad X(D) = \{y_D \times y_D\}$$

Then X is a 'set of sets' (a subobject of a power object), and clearly, in X, the sentence $\forall xy(x = y)$ is true. So X 'has at most one element'. We have the element relation $\in_{y_D}$ as a subobject of $\mathcal{P}(y_D) \times y_D$, and its restriction to a subobject of $X \times y_D$. This is the presheaf Z with $Z(E) = \emptyset$ and $Z(D) = \{(y_D \times y_D, \mathrm{id}_D), (y_D \times y_D, \beta)\}$. So we see that the sentence expressing 'every element of X has exactly two elements' is true. The presheaf $\bigcup X$ of 'elements of elements of X' is the presheaf $(\bigcup X)(E) = \emptyset$, $(\bigcup X)(D) = \{\mathrm{id}_D, \beta\}$ as subpresheaf of y_D. Now there cannot be any arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ from X to $\bigcup X$, because, in $X(D)$, the unique element is fixed by the action of β; however, in $(\bigcup X)(D)$ there is no fixed point for the action of β. Hence there is no 'choice function'.

10 Sheaves

In this chapter we shall generalize the notion of a 'sheaf on a topological space' to arbitrary categories.

First, let us recall what a sheaf on a topological space is. Let X be a space with set of opens $\mathcal{O}(X)$, considered as a poset with the inclusion order.

A *presheaf* F on X is simply a presheaf on $\mathcal{O}(X)$. So for two opens $U \subseteq V$ in X we have $F(U \subseteq V) : F(V) \to F(U)$, with the usual conditions. F is called *separated* if for any two elements x, y of $F(U)$ and any open cover $U = \bigcup_i U_i$ of U, if $F(U_i \subseteq U)(x) = F(U_i \subseteq U)(y)$ for all i, then $x = y$.

F is called a *sheaf*, if for every system of elements $x_i \in F(U_i)$, indexed by an open cover $U = \bigcup_i U_i$ of U, such that for every pair i, j we have $F(U_i \cap U_j \subseteq U_i)(x_i) = F(U_i \cap U_j \subseteq U_j)(x_j)$, there is a *unique* element $x \in F(U)$ such that $x_i = F(U_i \subseteq U)(x)$ for each i.

Such a system of elements x_i is called a *compatible family*, and x is called an *amalgamation* of it. The most common examples of sheaves on X are sheaves of partial functions: $F(U)$ is a set of functions $U \to Y$ (for example, continuous functions to a space Y), and $F(U \subseteq V)(f)$ is the restriction of f to U.

Example. Let X be $\mathbb{R}$ with the discrete topology; for $U \subseteq \mathbb{R}$ let $F(U)$ be the set of *injective* functions from U to $\mathbb{N}$. F is separated, but not a sheaf (check!).

In generalizing from $\mathcal{O}(X)$ to an arbitrary category $\mathcal{C}$, we see that what we lack is the notion of a 'cover'. Because $\mathcal{C}$ is in general not a preorder, it will not do to define a 'cover of an object C' as a collection of *objects* (as in the case of $\mathcal{O}(X)$); rather, a cover of C will be a *sieve* on C.

We shall denote the maximal sieve on C by $\max(C)$.

Definition 10.1 Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ specifies, for every object C of $\mathcal{C}$, a family $\mathrm{Cov}(C)$ of 'covering sieves' on C, in such a way that the following conditions are satisfied:

i) $\max(C) \in \mathrm{Cov}(C)$

ii) If $R \in \mathrm{Cov}(C)$ then for every $f : C' \to C$, $f^*(R) \in \mathrm{Cov}(C')$

iii) If R is a sieve on C and S is a covering sieve on C, such that for every arrow $f : C' \to C$ from S we have $f^*(R) \in \mathrm{Cov}(C')$, then $R \in \mathrm{Cov}(C)$

We note an immediate consequence of the definition:

Proposition 10.2 *a) If $R \in \mathrm{Cov}(C)$, S a sieve on C and $R \subseteq S$, then $S \in \mathrm{Cov}(C)$;*

b) If $R, S \in \mathrm{Cov}(C)$ then $R \cap S \in \mathrm{Cov}(C)$

Proof. For a), just observe that for every $f \in R$, $f^*(S) = \max(C')$; apply i) and iii) of 10.1. For b), note that if $f \in R$ then $f^*(S) = f^*(R \cap S)$, and apply ii) and iii). ∎

Definition 10.3 A *universal closure operation* on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ assigns to every presheaf X an operation $\overline{(\cdot)} : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ such that the following hold:

i) $A \le \bar{A}$

ii) $\bar{A} = \bar{\bar{A}}$

iii) $A \le B \Rightarrow \bar{A} \le \bar{B}$

iv) For $\phi : Y \to X$ and $A \in \mathrm{Sub}(X)$, $\phi^\sharp(\bar{A}) = \overline{\phi^\sharp(A)}$

Definition 10.4 A *Lawvere-Tierney topology* on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is an arrow $J : \Omega \to \Omega$ (where, as usual, Ω denotes the subobject classifier of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$), such that the following hold:

i) $R \subseteq J_C(R)$ for every sieve R on C

ii) $J_C(R \cap S) = J_C(R) \cap J_C(S)$

iii) $J_C(J_C(R)) = J_C(R)$

Theorem 10.5 *The following notions are equivalent (that is, each of them determines the others uniquely):*

1) A Grothendieck topology on $\mathcal{C}$

2) A Lawvere-Tierney topology on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$

3) A universal closure operation on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$

4) A full subcategory $\mathcal{E}$ *of* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$*, such that the inclusion* $\mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ *has a left adjoint which preserves finite limits*

Proof. We first prove the equivalence of the first three notions; the equivalence of these with notion 4 requires more work and is relegated to a separate proof.

1) $\Rightarrow$ 2). Given a Grothendieck topology Cov on $\mathcal{C}$, define $J : \Omega \to \Omega$ by

$$J_C(R) = \{h : C' \to C \mid h^*(R) \in \mathrm{Cov}(C')\}$$

If $h : C' \to C$ is an element of $J_C(R)$ and $g : C'' \to C'$ arbitrary, then $(hg)^*(R) = g^*(h^*(R)) \in \mathrm{Cov}(C'')$ so $hg \in J_C(R)$. Hence $J_C(R)$ is a sieve on C. Similarly, J is a natural transformation: for $f : C' \to C$ we have

$$\begin{array}{rcr} J_{C'}(f^*(R)) & = & \{h : C'' \to C' \mid h^*(f^*(R)) \in \mathrm{Cov}(C'')\} \\ & = & \{h : C'' \to C' \mid (fh)^*(R) \in \mathrm{Cov}(C'')\} \\ & = & \{h : C'' \to C' \mid fh \in J_C(R)\} \\ & = & f^*(J_C(R)) \end{array}$$

To prove $R \subseteq J_C(R)$ we just apply condition i) of 10.1, since $h^*(R) = \max(C')$ for $h \in R$.

By 10.2 we have $R \cap S \in \mathrm{Cov}(C)$ if and only if both $R \in \mathrm{Cov}(C)$ and $S \in \mathrm{Cov}(C)$, and together with the equation $h^*(R \cap S) = h^*(R) \cap h^*(S)$ this implies that $J_C(R \cap S) = J_C(R) \cap J_C(S)$.

Finally, since J preserves $\cap$, it preserves $\subseteq$ ($A \subseteq B$ iff $A \cap B = A$). We have proved $R \subseteq J_C(R)$, so $J_C(R) \subseteq J_C(J_C(R))$ follows. For the converse, suppose $h \in J_C(J_C(R))$ so $h^*(J_C(R)) \in \mathrm{Cov}(C')$. We need to prove $h^*(R) \in \mathrm{Cov}(C')$. Now for any $g \in h^*(J_C(R))$ we have $(hg)^*(R) \in \mathrm{Cov}(C'')$ so $g^*(h^*(R)) \in \mathrm{Cov}(C'')$. Hence by condition iii) of 10.1, $h^*(R) \in \mathrm{Cov}(C')$. This completes the proof that J is a Lawvere-Tierney topology.

2) $\Rightarrow$ 3). Suppose we are given a Lawvere-Tierney topology J. Define the operation $\bar{(\cdot)} : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$ as follows: if $A \in \mathrm{Sub}(X)$ is classified by $\phi : X \to \Omega$ then $\bar{A}$ is classified by $J\phi$. So

$$\bar{A}(C) = \{x \in X(C) \,|\, J_C(\phi_C(x)) = \max(C)\}$$

Then if $f : Y \to X$ is a map of presheaves and $A \in \mathrm{Sub}(X)$, both subobjects $f^\sharp(\bar{A})$ and $\overline{f^\sharp(A)}$ are classified by $J\phi f : Y \to \Omega$, hence they are equal. This proves iv) of definition 10.3. i) follows from condition i) of 10.4 and ii) from iii) of 10.4; finally, that $\bar{(\cdot)}$ is order-preserving follows from the fact that J_C preserves $\subseteq$.

3) $\Rightarrow$ 1). Given a universal closure operation $\bar{(\cdot)}$ on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, we define $\mathrm{Cov}(C) = \{R \in \Omega(C) \,|\, \bar{R} = y_C \text{ in } \mathrm{Sub}(y_C)\}$. Under the identification of sieves on C with subobjects of y_C, $f^*(R)$ corresponds to $(y_f)^\sharp(R)$. So from condition iv) in 10.3 it follows that if $R \in \mathrm{Cov}(C)$ and $f : C' \to C$, $f^*(R) \in \mathrm{Cov}(C')$. Condition i) ($\max(C) \in \mathrm{Cov}(C)$) follows from i) of 10.3.

To prove iii) of 10.1, suppose $R \in \Omega(C)$, $S \in \mathrm{Cov}(C)$ and for every $f : C' \to C$ in S we have $f^*(R) \in \mathrm{Cov}(C')$. So $\bar{S} = y_C$ and for all $f \in S$, $y_{C'} = \overline{f^*(R)} = (y_f)^\sharp(\bar{R}) = f^*(\bar{R})$. But that means that for all $f \in S$, $f \in \bar{R}$. So $S \subseteq \bar{R}$; hence by iii) of 10.3, $y_C = \bar{S} = \bar{\bar{R}}$; but $\bar{\bar{R}} = \bar{R}$ by ii) of 10.3, so $y_C = \bar{R}$, so $R \in \mathrm{Cov}(C)$, as desired. ∎

As said in the beginning of this proof, the equivalence of 4) with the other notions requires more work. We start with some definitions.

Definition 10.6 Let Cov be a Grothendieck topology on $\mathcal{C}$, and $\bar{(\cdot)}$ the associated universal closure operation on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

A presheaf F is *separated* for Cov if for each $C \in \mathcal{C}_0$ and $x, y \in F(C)$, if the sieve $\{f : C' \to C \,|\, F(f)(x) = F(f)(y)\}$ covers C, then $x = y$.

A subpresheaf G of F is *closed* if $\bar{G} = G$ in $\mathrm{Sub}(F)$.

A subpresheaf G of F is *dense* if $\bar{G} = F$ in $\mathrm{Sub}(F)$.

Exercise 161 i) A subpresheaf G of F is closed if and only if for each $x \in F(C)$: if $\{f : C' \to C \,|\, F(f)(x) \in G(C')\}$ covers C, then $x \in G(C)$

ii) F is separated if and only if the diagonal: $F \to F \times F$ is a closed subobject (this explains the term 'separated': in French, the word 'séparé' is synonymous with 'Hausdorff')

iii) A subpresheaf G of F is dense if and only if for each $x \in F(C)$, the sieve $\{f : C' \to C \mid F(f)(x) \in G(C')\}$ is covering

iv) A sieve R on C, considered as subobject of y_C, is dense if and only if it is covering

Definition 10.7 Let F be a presheaf, C an object of $\mathcal{C}$. A *compatible family* in F at C is a family $(x_f \mid f \in R)$ indexed by a sieve R on C, of elements $x_f \in F(\mathrm{dom}(f))$, such that for $f : C' \to C$ in R and $g : C'' \to C'$ arbitrary, $x_{fg} = F(g)(x_f)$. In other words, a compatible family is an arrow $R \to F$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. An *amalgamation* of such a compatible family is an element x of $F(C)$ such that $x_f = F(f)(x)$ for all $f \in R$. In other words, an amalgamation is an extension of the map $R \to F$ to a map $y_C \to F$.

Exercise 162 F is separated if and only if each compatible family in F, indexed by a covering sieve, has *at most one* amalgamation.

Definition 10.8 F is a *sheaf* if every compatible family in F, indexed by a covering sieve, has *exactly one* amalgamation.

Exercise 163 Suppose G is a subpresheaf of F. If G is a sheaf, then G is closed in $\mathrm{Sub}(F)$. Conversely, every closed subpresheaf of a sheaf is a sheaf.

Example. Let Y be a presheaf. Define a presheaf Z as follows: $Z(C)$ consists of all pairs (R, ϕ) such that $R \in \mathrm{Cov}(C)$ and $\phi : R \to Y$ is an arrow in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. If $f : C' \to C$ then $Z(f)(R, \phi) = (f^*(R), \phi f')$ where f' is such that

$$\begin{tikzcd} f^*(R) \arrow[r, "f'"] \arrow[d] & R \arrow[d] \\ y_{C'} \arrow[r, "y_f"'] & y_C \end{tikzcd}$$

is a pullback.

Suppose we have a compatible family in Z, indexed by a covering sieve S on C. So for each $f \in S$, $f : C' \to C$ there is $R_f \in \mathrm{Cov}(C')$, $\phi_f : R_f \to Y$, such that for $g : C'' \to C'$ we have that $R_{fg} = g^*(R_f)$ and $\phi_{fg} : R_{fg} \to Y$ is $\phi_f g'$ where $g' : R_{fg} \to R_f$ is the pullback of $y_g : y_{C''} \to y_{C'}$.

Then this family has an amalgamation in Z: define $T \in \mathrm{Cov}(C)$ by $T = \{fg \mid f \in S, g \in R_f\}$. T is covering since for every $f \in S$ we have $R_f \subseteq f^*(T)$. We can define $\chi : T \to Y$ by $\chi(fg) = \phi_f(g)$. So the presheaf Z satisfies the 'existence' part of the amalgamation condition for a sheaf. It does not in general satisfy the uniqueness part.

Exercise 164 Prove that F is a sheaf if and only if for every presheaf X and every dense subpresheaf A of X, any arrow $A \to F$ has a unique extension to an arrow $X \to F$.

The 'plus' construction. We define, for every presheaf X on $\mathcal{C}$, a presheaf X^+ as follows. $X^+(C)$ is the set of *equivalence classes* of pairs (R, ϕ) with $R \in \mathrm{Cov}(C)$ and $\phi : R \to X$, where $(R, \phi) \sim (S, \psi)$ holds if and only if there is a covering sieve T on C, such that $T \subseteq R \cap S$ and ϕ and ψ agree on T. Since $\mathrm{Cov}(C)$ is closed under intersections, this is evidently an equivalence relation.

The construction $(\cdot)^+$ is a functor $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$: for $f : X \to Y$ define $(f)^+ : X^+ \to Y^+$ by $(f)^+_C(R, \phi) = (R, f\phi)$. This is well-defined on equivalence classes.

We have a natural transformation ζ from the identity on $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ to $(\cdot)^+$: $(\zeta_X)_C(x) = (y_C, \phi)$ where $\phi : y_C \to X$ corresponds to x in the Yoneda Lemma ($\phi_C(\mathrm{id}_C) = x$).

The following lemma is 'by definition'.

Lemma 10.9 *Let X be a presheaf on $\mathcal{C}$.*

i) *X is separated if and only if ζ_X is mono.*

ii) *X is a sheaf if and only if ζ_X is an isomorphism.*

Lemma 10.10 *Let X be a presheaf, F a sheaf, $g : X \to F$ an arrow in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$*. Then g factors through $\zeta_X : X \to X^+$ via a unique $\tilde{g} : X^+ \to F$:*

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "\zeta_X"'] & & F \\ & X^+ \arrow[ur, dashed, "\tilde{g}"'] & \end{tikzcd}$$

Proof. For $[(R, \phi)] \in X^+(C)$, define $\tilde{g}_C([(R, \phi)])$ to be the unique amalgamation in $F(C)$ of the composite $g\phi : R \to F$. This is well-defined, for if $(R, \phi) \sim (S, \psi)$ then for some covering sieve $T \subseteq R \cap S$ we have that $g\phi$ and $g\psi$ agree on T; hence they have the same amalgamation. Convince yourself that $\tilde{g}$ is natural. By inspection, the diagram commutes, and $\tilde{g}$ is the unique arrow with this property. ■

Lemma 10.11 *For every presheaf X, X^+ is separated.*

Proof. For, suppose (R, ϕ) and (S, ψ) are representatives of elements of $X^+(C)$ such that, for some covering sieve T of C it holds that for each $f \in T$ there is a cover $T_f \subseteq F^*(R) \cap f^*(S)$ such that $\phi f'$ and $\psi f''$ agree on T_f, where f' and f'' are as in the pullback diagrams

$$\begin{tikzcd} f^*(R) \arrow[r, "f'"] \arrow[d] & R \arrow[d] & f^*(S) \arrow[r, "f''"] \arrow[d] & S \arrow[d] \\ y_{C'} \arrow[r, "f"'] & y_C & y_{C''} \arrow[r, "f"'] & y_C \end{tikzcd}$$

Let $U = \{fg \mid f \in T, g \in T_f\}$. Then U is a covering sieve on C, $U \subseteq R \cap S$, and ϕ and ψ agree on U. Hence $(R, \phi) \sim (S, \psi)$, so they represent the same element of $X^+(C)$. ■

Lemma 10.12 *If X is separated, X^+ is a sheaf.*

Proof. Suppose we have a compatible family in X^+, indexed by a covering sieve R on C. So for each $f : C' \to C$ in R we have (R_f, ϕ_f), $\phi_f : R \to X$.

In order to find an amalgamation, we define a sieve S and a map $\psi : S \to X$ by:

$$\begin{array}{rcl} S & = & \{fg \mid f \in R, g \in R_f\} \\ \psi(fg) & = & \phi_f(g) \end{array}$$

Certainly S is a covering sieve on C, but it is not a priori clear that ψ is well-defined. For it may be the case that for $f, f' \in R$, $g \in R_f$ and $g' \in R_{f'}$, $fg = f'g'$:

$$\begin{tikzcd} & C' \arrow[dr, "f"] & \\ C'' \arrow[ur, "g"] \arrow[dr, "g'"'] & & C \\ & D \arrow[ur, "f'"'] & \end{tikzcd}$$

We need to show that in this case, $\phi_f(g) = \phi_{f'}(g')$.

The fact that we have a compatible family means that $(g^*(R_f), \phi_f h) \sim ((g')^*(R_{f'}), \phi_{f'} h')$ in the equivalence relation defining $X^+(C'')$, where h and h' are as in the pullback diagrams

$$\begin{tikzcd} g^*(R_f) \arrow[r, "h"] \arrow[d] & R_f \arrow[d] & (g')^*(R_{f'}) \arrow[r, "h'"] \arrow[d] & R_{f'} \arrow[d] \\ y_{C''} \arrow[r, "y_g"'] & y_{C'} & y_{C''} \arrow[r, "y_{g'}"'] & y_D \end{tikzcd}$$

That means that there is a covering sieve T on C'' such that $T \subseteq g^*(R_f) \cap (g')^*(R_{f'})$ on which $\phi_f h$ and $\phi_{f'} h'$ coincide; hence, for all $k \in T$ we have that $X(k)(\phi_f(g)) = X(k)(\phi_{f'}(g'))$. Since X is separated by assumption, $\phi_f(g) = \phi_{f'}(g')$ as desired.

We have obtained an amalgamation. It is unique because X^+ is separated by lemma 10.11. ∎

The functor $\mathbf{a} : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is defined by applying $(\cdot)^+$ twice: $\mathbf{a}(X) = X^{++}$. By lemmas 10.11 and 10.12, $\mathbf{a}(X)$ is always a sheaf. There is a natural transformation η from the identity to $\mathbf{a}$ obtained by the composition $X \xrightarrow{\zeta} X^+ \xrightarrow{\zeta} X^{++} = \mathbf{a}(X)$; by twice applying lemma 10.10 one sees that every arrow from a presheaf X to a sheaf F factors uniquely through η. This exhibits η as the unit of an adjunction between $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and its full subcategory of sheaves. If we denote the latter by $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and regard $\mathbf{a}$ as a functor $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, then $\mathbf{a}$ is left adjoint to the inclusion functor.

The functor $\mathbf{a}$ is usually called *sheafification*, or the *associated sheaf* functor.

Lemma 10.13 *The sheafification functor preserves finite limits.*

Proof. It is enough to show that $(\cdot)^+$ preserves the terminal object and pullbacks. That $(\cdot)^+$ preserves 1 is obvious (1 is always a sheaf). Suppose

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "v"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & V \end{tikzcd}$$

is a pullback diagram. In order to show that also its $(\cdot)^+$-image is, suppose (R, ϕ) and (S, ψ) represent elements of $Y^+(C)$, $Z^+(C)$, respectively, such that $(R, f\phi) \sim (S, g\psi)$ in the equivalence relation defining $V^+(C)$. Then there is a covering sieve $T \subseteq R \cap S$ such that the square

$$\begin{tikzcd} T \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & V \end{tikzcd}$$

commutes. By the pullback property, there is a unique factoring map $\chi : T \to X$; that is, an element (T, χ) of $X^+(C)$ such that $[(R, \phi)] = u^+([(T, \chi)])$ and $[(S, \psi)] = v^+([(T, \chi)])$, which shows that the $(\cdot)^+$-image of the given diagram is a pullback. ■

We now wrap up to finish the proof of Theorem 10.5: every Grothendieck topology gives a category of sheaves, which is a full subcategory of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ such that the inclusion has a left adjoint **a** which preserves finite limits by 10.13.

Conversely, suppose such a full subcategory $\mathcal{E}$ of $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is given, with inclusion $i : \mathcal{E} \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and left adjoint $r : \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}} \to \mathcal{E}$, r preserving finite limits. This then determines a universal closure operation as follows. Given a subpresheaf A of X, define $\bar{A}$ as given by the pullback:

$$\begin{tikzcd} \bar{A} \arrow[r] \arrow[d] & ir(A) \arrow[d] \\ X \arrow[r, "\eta"'] & ir(X) \end{tikzcd}$$

where $\eta : X \to ir(X)$ is the unit of the adjunction $r \dashv i$. Since r preserves finite limits, ir preserves monos, so the map $\bar{A} \to X$ is monic. Writing down the naturality square for η for the inclusion $A \to X$ we see that this inclusion factors through $\bar{A} \to X$, so $A \leq \bar{A}$. It is immediate that $\bar{(\cdot)}$ is order-preserving; and because i is full and faithful so ri is naturally isomorphic to the identity on $\mathcal{E}$, ir is (up to isomorphism) idempotent, from which it is easy to deduce that $\bar{\bar{A}} = \bar{A}$. The final property of a universal closure operation, stability under pullback, follows again from the fact that r, and hence ir, preserves pullbacks. ■

10.1 Examples of Grothendieck topologies

1. As always, there are the two trivial extremes. The smallest Grothendieck topology (corresponding to the maximal subcategory of sheaves) has $\mathrm{Cov}(C)$ equal to $\{\max(C)\}$ for all C. The only dense subpresheaves are the maximal ones; every presheaf is a sheaf.

2. The other extreme is the biggest Grothendieck topology: $\mathrm{Cov}(C) = \Omega(C)$. Every subpresheaf is dense; the only sheaf is the terminal object 1.

3. Let X be a topological space with set of opens $\mathcal{O}(X)$, regarded as a category: a poset under the inclusion order. A sieve on an open set U can be identified with a downwards closed collection R of open subsets of U. The standard Grothendieck topology has $R \in \mathrm{Cov}(U)$ iff $\bigcup R = U$. Sheaves for this Grothendieck topology coincide with the familiar sheaves on the space X.

4. The *dense* or $\neg\neg$-topology is defined by:

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, \forall f : C' \to C \exists g : C'' \to C' \,(fg \in R)\}$$

This topology corresponds to the Lawvere-Tierney topology $J : \Omega \to \Omega$ defined by

$$J_C(R) \;=\; \{h : C' \to C \,|\, \forall f : C'' \to C' \exists g : C''' \to C'' \,(hfg \in R)\}$$

This topology has the property that for every sheaf F, the collection of subsheaves of F forms a Boolean algebra.

5. For this example we assume that in the category $\mathcal{C}$, every pair of arrows with common codomain fits into a commutative square. Then the *atomic* topology takes all *nonempty* sieves as covers. This corresponds to the Lawvere-Tierney topology

$$J_C(R) \;=\; \{h : C' \to C \,|\, \exists f : C'' \to C' \,(hf \in R)\}$$

This topology has the property that for every sheaf F, the collection of subsheaves of F forms an *atomic* Boolean algebra: an *atom* in a Boolean algebra is a minimal non-bottom element. An atomic Boolean algebra is such that for every non-bottom x, there is an atom which is $\leq x$.

6. Let U be a subpresheaf of the terminal presheaf 1. With U we can associate a set of objects $\tilde{U}$ of $\mathcal{C}$ such that whenever $f : C' \to C$ is an arrow and $C \in \tilde{U}$, then $C' \in \tilde{U}$. Namely, $\tilde{U} = \{C \,|\, U(C) \neq \emptyset\}$. To such U corresponds a Grothendieck topology, the *open topology* determined by U, given by

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, \forall f : C' \to C \,(C' \in \tilde{U} \Rightarrow f \in R)\}$$

and associated Lawvere-Tierney topology

$$J_C(R) \;=\; \{h : C' \to C \,|\, \forall f : C'' \to C' \,(C'' \in \tilde{U} \Rightarrow hf \in R)\}$$

Let $\mathcal{D}$ be the full subcategory of $\mathcal{C}$ on the objects in $\tilde{U}$. Then there is an equivalence of categories between $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and $\mathrm{Set}^{\mathcal{D}^{\mathrm{op}}}$.

7. For U and $\tilde{U}$ as in the previous example, there is also the *closed* topology determined by U, given by

$$\mathrm{Cov}(C) \;=\; \{R \in \Omega(C) \,|\, C \in \tilde{U} \text{ or } R = \max(C)\}$$

There is an equivalence between $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and the category of presheaves on the full subcategory of $\mathcal{C}$ on the objects *not* in $\tilde{U}$.

Exercise 165 We consider the category $\mathcal{C}$ whose objects are subsets of $\mathbb{N}$, and arrows $A \to B$ are *finite-to-one* functions, i.e. functions f satisfying the requirement that for every $b \in B$, the set $\{a \in A \,|\, f(a) = b\}$ is finite.

a) Show that $\mathcal{C}$ has pullbacks.

b) Define for every object A of $\mathcal{C}$ a set $\mathrm{Cov}(A)$ of sieves on A as follows: $R \in \mathrm{Cov}(A)$ if and only if R contains a finite family $\{f_1, \ldots, f_n\}$ of functions into A, which is *jointly almost surjective*, that is: the set

$$A - \bigcup_{i=1}^{n} \mathrm{Im}(f_i)$$

is finite.

Show that Cov is a Grothendieck topology.

c) Show that if $R \in \mathrm{Cov}(A)$, then R contains a family $\{f_1, \ldots, f_n\}$ which is jointly almost surjective and moreover, every f_i is injective.

d) Given a nonempty set X and an object A of $\mathcal{C}$, we define $F_X(A)$ as the set of equivalence classes of functions $\xi : A \to X$, where $\xi \sim \eta$ if $\xi(n) = \eta(n)$ for all but finitely many $n \in A$.

Show that this definition can be extended to the definition of a presheaf F_X on $\mathcal{C}$.

e) Show that F_X is a sheaf for Cov.

10.2 Structure of the category of sheaves

In this section we shall see, among other things, that also the category $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is a topos.

Proposition 10.14 $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is closed under arbitrary limits in* $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$.

Proof. This is rather immediate from the defining property of sheaves and the way (point-wise) limits are calculated in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Suppose $F : I \to \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ is a diagram of sheaves with limiting cone $(X, (\mu_i : X \to F(i)))$ in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. We show that X is a sheaf.

Suppose $R \in \mathrm{Cov}(C)$ and $\phi : R \to X$ is a map in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$. Since every $F(i)$ is a sheaf, every composite $\mu_i\phi : R \to F(i)$ has a unique amalgamation $y_i \in F(i)(C)$, and by uniqueness these satisfy, for every map $k : i \to j$ in the index category I, the equality $(F(k))_C(y_i) = y_j$. Since $X(C)$ is the vertex of a limiting cone for the diagram $F(\cdot)(C) : I \to \mathrm{Set}$, there is a unique $x \in X(C)$ such that $(\mu_i)_C(x) = y_i$ for each i. But this means that x is an amalgamation (and the unique such) for $R \xrightarrow{\phi} X$. ∎

Proposition 10.15 *Let X be a presheaf, Y a sheaf. Then Y^X is a sheaf.*

Proof. Suppose $A \to Z$ is a dense subobject, and $A \xrightarrow{\phi} Y^X$ a map. By exercise 164 we have to see that ϕ has a unique extension to a map $Z \to Y^X$. Now ϕ transposes to a map $\tilde{\phi} : A \times X \to Y$. By stability of the closure operation, if $A \to Z$ is dense then so is $A \times X \to Z \times X$. Since Y is a sheaf, $\tilde{\phi}$ has a unique extension to $\psi : Z \times X \to Y$. Transposing back gives $\bar{\psi} : Z \to Y^X$, which is the required extension of ϕ. ∎

Corollary 10.16 *The category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is cartesian closed.*

Now we turn to the subobject classifier in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Let $J : \Omega \to \Omega$ be the associated Lawvere-Tierney topology. Sieves on C which are in the image of J_C are called *closed*. This is good terminology, since a closed sieve on C is the same thing as a closed subpresheaf of y_C.

By exercise 163 we know that subsheaves of a sheaf are the closed subpresheaves, and from exercise 161i) we know that a subpresheaf is closed if and only if its classifying map takes values in the image of J. This is a subobject of Ω; let us call it Ω_J. So subobjects in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ admit unique classifying maps into Ω_J; note that the map $1 \xrightarrow{t} \Omega$, which picks out the maximal sieve on any C, factors through Ω_J since every maximal sieve is closed. So $1 \xrightarrow{t} \Omega_J$ is a subobject classifier in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ provided we can show that it is a map between sheaves. It is easy to see (and a special case of 10.14) that 1 is a sheaf. For Ω_J this requires a little argument.

Proposition 10.17 *The presheaf Ω_J is a sheaf.*

Proof. We have seen that the arrow $1 \xrightarrow{t} \Omega_J$ classifies closed subobjects. Therefore, in order to show that Ω_J has the unique-extension property w.r.t. dense inclusions, it is enough to see that whenever X is a dense subpresheaf of Y there is a bijection between the closed subpresheaves of X and the closed subpresheaves of Y.

For a closed subpresheaf A of X let $k(A)$ be the closure of A in $\mathrm{Sub}(Y)$. For a closed subpresheaf B of Y let $l(B) = B \cap X$; this is a closed subpresheaf of X.

Now $kl(B) = k(B \cap X) = \overline{B \cap X} = \bar{B} \cap \bar{X} = \bar{B} = B$ since X is dense and B closed. Conversely, $lk(A) = \bar{A} \cap X$ which is (by stability of closure) the closure of A in X. But A was closed, so this is A. Hence the maps k and l are inverse to each other, which finishes the proof. ∎

Corollary 10.18 *The category* $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ *is a topos.*

Definition 10.19 A pair $(\mathcal{C}, \mathrm{Cov})$ of a small category and a Grothendieck topology on it is called a *site*. For a sheaf on $\mathcal{C}$ for Cov, we also say that it is a *sheaf on the site* $(\mathcal{C}, \mathrm{Cov})$. A *Grothendieck topos* is a category of sheaves on a site.

Not every topos is a Grothendieck topos. For the moment, there is only one simple example to give of a topos that is not Grothendieck: the category of finite sets. It is not a Grothendieck topos, for example because it does not have all small limits.

Exercise 166 The terminal category **1** is a topos. Is it a Grothendieck topos?

Let us say something about power objects and the natural numbers in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$.

For power objects there is not much more to say than this: for a sheaf X, its power object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is Ω_J^X; we shall also write $\mathcal{P}_J(X)$. By the Yoneda Lemma we have a natural 1-1 correspondence between $\mathcal{P}_J(X)(C)$ and the set of closed subpresheaves of $y_C \times X$; for $f : C' \to C$ and A a closed subpresheaf of $y_C \times X$, $\mathcal{P}_J(X)(f)(A)$ is given by $(y_f \times \mathrm{id}_X)^\sharp(A)$.

Next, let us discuss natural numbers. We use exercise 156 which says that the constant presheaf with value $\mathbb{N}$ is a natural numbers object in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, and we also use the following result:

Exercise 167 Suppose $\mathcal{E}$ has a natural numbers object and $F : \mathcal{E} \to \mathcal{F}$ is a functor which has a right adjoint and preserves the terminal object. Then F preserves the natural numbers object.

So the natural numbers object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ is N^{++}, where N is the constant presheaf with value $\mathbb{N}$. In fact, we don't have to apply the 'plus' construction twice, because N is 'almost' separated: clearly, if n, m are two distinct natural numbers and $R \in \mathrm{Cov}(C)$ is such that for all $f \in R$ we have $nf = mf$, then $R = \emptyset$. So the only way that N can fail to be separated is that for some objects C we have $\emptyset \in \mathrm{Cov}(C)$. Now define the presheaf N' as follows:

$$N'(C) = \begin{cases} \mathbb{N} & \text{if } \emptyset \notin \mathrm{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \mathrm{Cov}(C) \end{cases}$$

Exercise 168 Prove:

a) N' is separated

b) $\zeta_N : N \to N^+$ factors through N'

c) $N^{++} \simeq (N')^+$

Colimits in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ are calculated as follows: take the colimit in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$, then apply the associated sheaf functor. For coproducts of sheaves, we have a simplification comparable to that of N. We write $\bigsqcup$ for the coproduct in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ and $\bigsqcup_J$ for the coproduct in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. So $\bigsqcup_J F_i = \mathbf{a}(\bigsqcup F_i)$, but if we define $\bigsqcup' F_i$ by

$$(\bigsqcup{}' F_i)(C) \;=\; \begin{cases} \bigsqcup F_i(C) & \text{if } \emptyset \notin \mathrm{Cov}(C) \\ \{*\} & \text{if } \emptyset \in \mathrm{Cov}(C) \end{cases}$$

then it is not too hard to show that $\bigsqcup_J F_i \simeq (\bigsqcup' F_i)^+$. Concretely, a compatible family in $\bigsqcup' F_i$ indexed by a covering sieve R on C, i.e./ a map $\phi : R \to \bigsqcup' F_i$, gives for each i a sub-sieve R_i and a map $\phi_i : R_i \to F_i$. The system of subsieves R_i has the property that if $h : C' \to C$ is an element of $R_i \cap R_j$ and $i \neq j$, then $\emptyset \in \mathrm{Cov}(C')$. Of course, such compatible families are still subject to the equivalence relation defining $(\bigsqcup' F_i)^+$.

Exercise 169 Prove:

i) Coproducts are stable in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$

ii) For any sheaf F, $F^{N_J} \simeq \prod_{n\in\mathbb{N}} F$

Images in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$: given a map $\phi : F \to G$ between sheaves, the image of ϕ (as subsheaf of G) is the closure of the image in $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of the same map. The arrow ϕ is an epimorphism in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ if and only if for each C and each $x \in G(C)$, the sieve $\{f : C' \to C \mid \exists y \in F(C')(\phi_{C'}(y) = xf)\}$ covers C.

Exercise 170 Prove this characterization of epis in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Prove also that in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$, an arrow which is both mono and epi is an isomorphism.

Regarding the structure of the lattice of subobjects in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ of a sheaf F, we know that these are the closed subpresheaves, so the fixed points of the closure operation. That the subobjects again form a Heyting algebra is then a consequence of the following exercise.

Exercise 171 Suppose H is a Heyting algebra with operations $\bot, \top, \wedge, \vee, \to$ and let $j : H \to H$ be order-preserving, idempotent, inflationary (that is: $x \leq j(x)$ for all $x \in H$), and such that $j(x \wedge y) = j(x) \wedge j(y)$. Let H_j be the set of fixed points of j. Then H_j is a Heyting algebra with operations:

$$\begin{array}{cc} \top_j = \top & \bot_j = j(\bot) \\ x \wedge_j y = x \wedge y & x \vee_j y = j(x \vee y) \\ & x \to_j y = x \to y \end{array}$$

Exercise 172 If H is a Heyting algebra, show that the map $\neg\neg : x \mapsto (x \to \bot) \to \bot$ satisfies the requirements of the map j in exercise 171. Show also that $H_{\neg\neg}$ is a Boolean algebra.

Exercise 173 Let J be the Lawvere-Tierney topology corresponding to the dense topology (see section 10.1). Show that in the Heyting algebra $\Omega(C)$, J_C is the map $\neg\neg$ of exercise 172.

As for presheaves, we can express the interpretation of first-order languages in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ in terms of a 'forcing' definition. The basic setup is the same; only now, of course, we take sheaves as interpretation of the sorts, and closed subpresheaves (subsheaves) as interpretation of the relation symbols. We then define $[\![\,\varphi\,]\!]$ as a subsheaf of $[\![\,FV(\varphi)\,]\!]$ and let $\{\varphi\} : [\![\,FV(\varphi)\,]\!] \to \Omega_J$ be its classifying map. The notation $C \Vdash_J \varphi(a_1, \ldots, a_n)$ again means that $\{\varphi\}_C(a_1, \ldots, a_n)$ is the maximal sieve on C. This relation then again admits a definition by recursion on the formula φ. The inductive clauses of the definition of $\Vdash_J$ are the same for $\Vdash$ for the cases: atomic formula, $\wedge$, $\to$ and $\forall$, and we put:

- $C \Vdash_J \neg\varphi(a_1, \ldots, a_n)$ if and only if for every arrow $g : D \to C$ in $\mathcal{C}$ we have: if $D \Vdash_J \varphi(a_1 g, \ldots, a_n g)$ then $\emptyset$ covers D;
- $C \Vdash_J (\varphi \vee \psi)(a_1, \ldots, a_n)$ if and only if the sieve $\{g : C' \to C \,|\, C' \Vdash_J \varphi(a_1 g, \ldots, a_n g)$ or $C' \Vdash_J \psi(a_1 g, \ldots, a_n g)\}$ covers C;
- $C \Vdash_J \exists x \varphi(x, a_1, \ldots, a_n)$ if and only if the sieve $\{g : C' \to C \,|\, \exists x \in F(C')\, C' \Vdash_J \varphi(x, a_1 g, \ldots, a_n g)\}$ covers C (where F is the interpretation of the sort of x).

That this works should be no surprise in view of our characterisation of images in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ and our treatment of the Heyting structure on the subsheaves of a sheaf. We have the following properties of the relation $\Vdash_J$:

Theorem 10.20 *i) If $C \Vdash_J \varphi(a_1, \ldots, a_n)$ then for each arrow $f : C' \to C$, $C' \Vdash_J \varphi(a_1 f, \ldots, a_n f)$;*

ii) if R is a covering sieve on C and for every arrow $f : C' \to C$ in R we have $C' \Vdash_J \varphi(a_1 f, \ldots, a_n f)$, then $C \Vdash_J \varphi(a_1, \ldots, a_n)$.

Exercise 174 Let N_J be the natural numbers object in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$. Prove the same result as we had in exercise 156, that is: for the standard interpretation of te language of arithmetic in N_J, a sentence is true if and only it is true in the (classical) standard model of natural numbers.

Exercise 175 We assume that we have a site $(\mathcal{C}, \mathrm{Cov})$ and an object I of $\mathcal{C}$ satisfying the following conditions:

i) $\emptyset \notin \mathrm{Cov}(I)$

ii) If there is no arrow $I \to A$ then $\emptyset \in \mathrm{Cov}(A)$

iii) If there is an arrow $I \to A$ then every arrow $A \to I$ is split epi

We call a sheaf F in $\mathrm{Sh}(\mathcal{C}, \mathrm{Cov})$ $\neg\neg$-*separated* if for every object A of $\mathcal{C}$ and all $x, y \in F(A)$,

$$A \Vdash_J \neg\neg(x = y) \to x = y$$

Prove that the following two assertions are equivalent, for a sheaf F:

a) F is $\neg\neg$-separated

b) For every object A of $\mathcal{C}$ and all $x, y \in F(A)$ the following holds: if for every arrow $\phi : I \to A$ we have $x\phi = y\phi$ in $F(I)$, then $x = y$

10.3 Application: a model for the independence of the Axiom of Choice

In this section we treat a model, due to P. Freyd, which shows that in toposes where *classical* logic always holds, the axiom of choice need not be valid. Specifically, we construct a topos $\mathcal{F} = \mathrm{Sh}(F, \mathrm{Cov})$ and in $\mathcal{F}$ a subobject E of $N_J \times \mathcal{P}_J(N_J)$ with the properties:

i) $\mathcal{F}$ is *Boolean*, that is: every subobject lattice is a Boolean algebra;

ii) $\Vdash_J \forall n \exists \alpha((n, \alpha) \in E)$

iii) $\Vdash \neg\exists f \in \mathcal{P}_J(N_J)^{N_J} \forall n\,((n, f(n)) \in E)$

So, E is an N_J-indexed collection of nonempty (in a strong sense) subsets of $\mathcal{P}_J(N_J)$, but admits no choice function.

Let $\mathbb{F}$ be the following category: it has objects $\bar{n}$ for each natural number n, and an arrow $f : \bar{m} \to \bar{n}$ is a function $\{0, \ldots, m\} \to \{0, \ldots, n\}$ such that $f(i) = i$ for every i with $0 \leq i \leq n$. It is understood that there are no morphisms $\bar{m} \to \bar{n}$ for $m < n$. Note, that $\bar{0}$ is a terminal object in this category.

On $\mathbb{F}$ we let Cov be the dense topology, so a sieve R on $\bar{m}$ covers $\bar{m}$ if and only if for every arrow $g : \bar{n} \to \bar{m}$ there is an arrow $h : \bar{k} \to \bar{n}$ such that $gh \in R$. We shall work in the topos $\mathcal{F} = \mathrm{Sh}(\mathbb{F}, \mathrm{Cov})$, the *Freyd topos*. Let E_n be the object $\mathbf{a}(y_{\bar{n}})$, the sheafification of the representable presheaf on $\bar{n}$.

Lemma 10.21 Cov *has the following properties:*

a) Every covering sieve is nonempty

b) Every nonempty sieve on $\bar{0}$ is a cover

c) Every representable presheaf is separated

d) $y_{\bar{0}}$ has only two closed subobjects

Proof. For a), apply the definition of 'R covers $\bar{m}$' to the identity on $\bar{m}$; it follows that there is an arrow $h : \bar{k} \to \bar{m}$ such that $h \in R$.

For b), suppose S is a sieve on $\bar{0}$ and $\bar{k} \xrightarrow{f} \bar{0}$ is in S. Since $\bar{0}$ is terminal, for any $\bar{m} \xrightarrow{g} \bar{0}$ and any maps $\overline{m+k} \to \bar{k}$, $\overline{m+k} \to \bar{m}$, the square

$$\begin{tikzcd}
\overline{m+k} \arrow[r] \arrow[d] & \bar{k} \arrow[d, "f"] \\
\bar{m} \arrow[r, "g"'] & \bar{0}
\end{tikzcd}$$

commutes, so for any such g there is an h with $gh \in R$, hence R covers $\bar{0}$.

For c), suppose $g, g' : \bar{k} \to \bar{n}$ are such that for a cover R of $\bar{k}$ we have $gf = g'f$ for all $f \in R$. We need to see that $g = g'$. Pick $i \leq k$. Let $h : \overline{k+1} \to \bar{k}$ be such that $h(k+1) = i$. Since R covers $\bar{k}$ there is $u : \bar{l} \to \overline{k+1}$ such that $hu \in R$. Then $ghu = g'hu$, which means that $g(i) = ghu(k+1) = g'hu(k+1) = g'(i)$. So $g = g'$, as desired.

Finally, d) follows directly from b): suppose R is a closed sieve on $\bar{0}$. If $R \neq \emptyset$, then R is covering by b), hence (being also closed) equal to $\max(\bar{0})$. Hence the only closed sieves are $\emptyset$ and $\max(\bar{0})$. ∎

Proposition 10.22 *The unique map $E_n \to 1$ is an epimorphism.*

Proof. By lemma 10.21d), $1 = \mathbf{a}(y_{\bar{0}})$ has only two subobjects and $y_{\bar{n}}$ is nonempty, so the image of $E_n \to 1$ is 1. ∎

Proposition 10.23 *If $n > m$ then $E_n(\bar{m}) = \emptyset$.*

Proof. Since $y_{\bar{n}}$ is separated by 10.21c), $E_n = (y_{\bar{n}})^+$, so $E_n(\bar{m})$ is an equivalence class of morphisms $\tau : S \to y_{\bar{n}}$ in $\mathrm{Set}^{\mathbb{F}^{\mathrm{op}}}$, for a cover S of $\bar{m}$. We claim that such τ don't exist.

For, since such S is nonempty (10.21a)), pick $s : \bar{k} \to \bar{m}$ in S and let $f = \tau_{\bar{k}}(s)$, so $f : \bar{k} \to \bar{n}$. Let $g, h : \overline{k+1} \to \bar{k}$ be such that $g(k+1) = n$, and $h(k+1) = s(n) \leq m < n$. Then $sg = sh$ (check!). So

$$fg = \tau_{\bar{k}}(s)g = \tau_{\overline{k+1}}(sg) = \tau_{\overline{k+1}}(sh) = \tau_{\bar{k}}(s)h = fh$$

However, $fg(k+1) = f(n) = n$, whereas $fh(k+1) = f(s(n+1)) = s(n)$. Contradiction. ∎

Corollary 10.24 *The product sheaf $\prod_{n \in \mathbb{N}} E_n$ is empty.*

Proof. For, if $(\prod_n E_n)(\bar{m}) \neq \emptyset$ then by applying the projection $\prod_n E_n \to E_{\overline{m+1}}$ we would have $E_{m+1}(\bar{m}) \neq \emptyset$, contradicting 10.23. ∎

Proposition 10.25 *For each n there is a monomorphism $E_n \to \mathcal{P}_J(N_J)$.*

Proof. Since $E_n = \mathbf{a}(y_{\bar{n}})$ and $\mathcal{P}_J(N_J)$ is a sheaf, it is enough to construct a monomorphism $y_{\bar{n}} \to \mathcal{P}_J(N_J)$, which gives then a unique extension to a map from E_n; since $\mathbf{a}$ preserves monos, the extension will be mono if the given map is.

Fix n for the rest of the proof. Let $(g_k)_{k \in \mathbb{N}}$ be a 1-1 enumeration of all the arrows in $\mathbb{F}$ with codomain $\bar{n}$. For each g_i, let C_i be the smallest closed sieve on $\bar{n}$ containing g_i (i.e., C_i is the $J_{\bar{n}}$-image of the sieve generated by g_i).

$\mathcal{P}_J(N_J)(\bar{m})$ is the set of closed subpresheaves of $y_{\bar{m}} \times N_J$. Elements of $(y_{\bar{m}} \times N_J)(\bar{k})$ are pairs $(h, (S_i)_{i \in \mathbb{N}})$ where $h : \bar{k} \to \bar{m}$ and $(S_i)_i$ is an $\mathbb{N}$-indexed collection of sieves on $\bar{k}$, such that $S_i \cap S_j = \emptyset$ for $i \neq j$, and $\bigcup_i S_i$ covers $\bar{k}$.

Define $\mu_{\bar{m}} : y_{\bar{n}}(\bar{m}) \to \mathcal{P}_J(N_J)(\bar{m})$ as follows. For $f : \bar{m} \to \bar{n}$, $\mu_{\bar{m}}(f)$ is the subpresheaf of $y_{\bar{m}} \times N_J$ given by: $(h, (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k})$ iff for each i, $S_i \subseteq (fh)^*(C_i)$. It is easily seen that $\mu_{\bar{m}}(f)$ is a closed subpresheaf of $y_{\bar{m}} \times N_J$.

Let us first see that μ is a natural transformation. Suppose $g : \bar{l} \to \bar{m}$. For $h' : \bar{k} \to \bar{l}$ we have:

$$\begin{array}{ll} & (h', (S_i)_i) \in (y_g \times \mathrm{id}_{N_J})^\sharp(\mu_{\bar{m}}(f))(\bar{k}) \\ \text{iff} & (gh', (S_i)_i) \in \mu_{\bar{m}}(f)(\bar{k}) \\ \text{iff} & \forall i (\, S_i \subseteq (fgh')^*(C_i)) \\ \text{iff} & (h', (S_i)_i) \in \mu_{\bar{l}}(fg)(\bar{k}) \end{array}$$

Next, let us prove that μ is mono. Suppose $\mu_{\bar{m}}(f) = \mu_{\bar{m}}(f')$ for $f, f' : \bar{m} \to \bar{n}$. Let j and j' be such that in our enumeration, $f = g_j$ and $f' = g_{j'}$. Now consider the pair $\xi = (\mathrm{id}_{\bar{m}}, (S_i)_i)$, where S_i is the empty sieve if $i \neq j$, and $S_j = \max(\bar{m})$. Then ξ is easily seen to be an element of $\mu_{\bar{m}}(f)(\bar{m})$, so it must also be an element of $\mu_{\bar{m}}(f')(\bar{m})$, which means that $f' \in C_j$. So $C_j \cap C_{j'} \neq \emptyset$. But this means that we must have a commutative square in $\mathbb{F}$:

$$\begin{tikzcd} \bar{l} \arrow[r] \arrow[d] & \bar{m} \arrow[d, "f'"] \\ \bar{m} \arrow[r, "f"'] & \bar{n} \end{tikzcd}$$

It is easy to conclude from this that $f = f'$. ■

10.4 Application: a model for "every function from reals to reals is continuous"

In 1924, L.E.J. Brouwer published a paper: *Beweis, dass jede volle Funktion gleichmässig stetig ist* (Proof, that every total function is uniformly continuous), *Nederl. Akad. Wetensch. Proc.* **27**, pp.189–193. His lucubrations on intuitionistic mathematics had led him to the conclusion that every function from $\mathbb{R}$ to $\mathbb{R}$ must be continuous. Among present-day researchers of constructive mathematics, this statement is known as *Brouwer's Principle* (although die-hard intuitionists still refer to it as Brouwer's Theorem).

The principle can be made plausible in a number of ways; one is, to look at the reals from a computational point of view. If a computer, which can only deal with finite approximations of reals, computes a function, then for every required precision for $f(x)$ it must be able to approximate x closely enough and from there calculate $f(x)$ within the prescribed precision; this just means that f must be continuous.

In this section we shall show that the principle is *consistent* with higher-order intuitionistic type theory, by exhibiting a topos in which it holds, for the standard real numbers. In order to do this, we have of course to say what the "object of real numbers" in a topos is. That will be done in the course of the construction.

We shall work with a full subcategory $\mathbb{T}$ of the category Top of topological spaces and continuous functions. It doesn't really matter so much what $\mathbb{T}$ exactly is, but we require that:

- $\mathbb{T}$ is closed under finite products and open subspaces
- $\mathbb{T}$ contains the space $\mathbb{R}$ (with the euclidean topology)

We specify a Grothendieck topology on $\mathbb{T}$ by defining, for an object T of $\mathbb{T}$, that a sieve R on T covers T, if the set of open subsets U of T for which the inclusion $U \to T$ is in R, forms an open covering of T. It is easy to verify that this is a Grothendieck topology.

The first thing to note is that for this topology (we call it Cov), every representable presheaf is a sheaf, because it is a presheaf of (continuous) functions: given a compatible family $R \to y_T$ for R a covering sieve on X, this family contains maps $f_U : U \to T$ for every open U contained in a covering of X; and these maps agree on intersections, because we have a sieve. So they have a unique amalgamation to a continuous map $f : X \to T$, i.e. an element of $y_T(X)$.

Also for spaces S not necessarily in the category $\mathbb{T}$ we have sheaves $y_S = \mathrm{Cts}(-, S)$.

Recall that the Yoneda embedding preserves existing exponents in $\mathbb{T}$. This also extends to exponents which exist in Top but are not in $\mathbb{T}$. If T is a locally compact space, then for any space X we have an exponent X^T in Top: it is the set of continuous functions $T \to X$, equipped with the compact-open topology (a subbase for this topology is given by the sets $\mathcal{C}(C, U)$ of those continuous functions that map C into U, for a compact subset C of T and an open subset U of X). Thus, even if X is not an object of $\mathbb{T}$, we still have in $\mathrm{Sh}(\mathbb{T}, \mathrm{Cov})$:

$$y_{X^T} \simeq (y_X)^{(Y_T)}$$

Exercise 176 Prove this fact.

From now on, we shall denote the category $\mathrm{Sh}(\mathbb{T}, \mathrm{Cov})$ by $\mathcal{T}$.
Notation: in this section we shall dispense with all subscripts $(\cdot)_J$, since we shall only work in $\mathcal{T}$. So, N denotes the *sheaf* of natural numbers, $\mathcal{P}(X)$ is the power *sheaf* of X, $\Vdash$ refers to forcing in sheaves, etc.

The natural numbers are given by the constant sheaf N, the $\mathbb{N}$-fold coproduct of copies of 1. The rational numbers are formed as a quotient of $N \times N$ by an equivalence relation which can be defined in a quantifier-free way, and hence is also a constant sheaf; therefore the object of rational numbers Q is the constant sheaf on the classical rational numbers $\mathbb{Q}$, and therefore the $\mathbb{Q}$-fold coproduct of copies of 1.

Proposition 10.26 *In $\mathcal{T}$, N and Q are isomorphic to the representable sheaves $y_{\mathbb{N}}$, $y_{\mathbb{Q}}$ respectively, where $\mathbb{N}$ and $\mathbb{Q}$ are endowed with the discrete topology.*

Proof. We shall do this for N; the proof for Q is similar. An element of $y_{\mathbb{N}}(X)$ is a continuous function from X to the discrete space $\mathbb{N}$; this is the same thing as an open covering $\{U_n \mid n \in \mathbb{N}\}$ of pairwise disjoint sets; which in turn is the same thing as an (equivalence class of an) $\mathbb{N}$-indexed collection $\{R_n \mid n \in \mathbb{N}\}$ of sieves on X such that whenever for $n \neq m$, $f : Y \to X$ is in $R_n \cap R_m$, $Y = \emptyset$;

and moreover the sieve $\bigcup_n R_n$ covers X. But that last thing is just an element of $(\bigsqcup_n 1)(X)$. ∎

Under this isomorphism, the order on N and Q corresponds to the pointwise ordering on functions.

Exercise 177 Show that in $\mathcal{T}$, the objects N and Q are linearly ordered, that is: for every space X in $\mathbb{T}$, $X \Vdash \forall rs \in Q\,(r < s \vee r = r \vee s < r)$.

We now construct the *object of Dedekind reals* R_d. Just as in the classical definition, a real number is a Dedekind cut of rational numbers, that is: a pair (L, R) of subsets of Q satisfying:

i) $\forall q \in Q \neg (q \in L \wedge q \in R)$

ii) $\exists q (q \in L) \wedge \exists r (r \in R)$

iii) $\forall qr (q < r \wedge r \in L \to q \in L) \;\wedge\; \forall st (s < t \wedge s \in R \to t \in R)$

iv) $\forall q \in L \exists r (q < r \wedge r \in L) \;\wedge\; \forall s \in R \exists t (t < s \wedge t \in R)$

v) $\forall qr (q < r \to q \in L \vee r \in R)$

Write $\mathrm{Cut}(L, R)$ for the conjunction of these formulas. So the object of reals R_d is the subsheaf of $\mathcal{P}(Q) \times \mathcal{P}(Q)$ given by:

$$R_d(X) \;=\; \{(L, R) \in (\mathcal{P}(Q) \times \mathcal{P}(Q))(X) \,|\, X \Vdash \mathrm{Cut}(L, R)\}$$

This is always a sheaf, by theorem 10.20ii).

Proposition 10.27 *The sheaf R_d is isomorphic to the representable sheaf $y_{\mathbb{R}}$.*

Proof. Let W be an object of $\mathbb{T}$ and $(L, R) \in R_d(W)$. Then L and R are subsheaves of $y_W \times Q$, which is isomorphic to $y_{W \times \mathbb{Q}}$. So both L and R consist of pairs of maps (α, p) with $\alpha : Y \to W$, $p : Y \to \mathbb{Q}$ continuous. Since L and R are sub*sheaves* we have: if $(\alpha, p) \in L(Y)$ then for any $f : V \to Y$, $(\alpha f, pf) \in L(V)$, and if $(\alpha \restriction V_i, p \restriction V_i) \in L(V_i)$ for an open cover $\{V_i\}_i$ of Y, then $(\alpha, p) \in L(Y)$ (and similar for R, of course).

Now for such $(L, R) \in \mathcal{P}(\mathbb{Q})(W) \times \mathcal{P}(\mathbb{Q})(W)$ we have $(L, R) \in R_d(W)$ if and only if $W \Vdash \mathrm{Cut}(L, R)$. We are now going to spell out what this means, and see that such (L, R) uniquely determine a continuous function $W \to \mathbb{R}$.

i)′ For $\beta : W' \to W$ and $q : W' \to \mathbb{Q}$, not both $(\beta, q) \in L(W')$ and $(\beta, q) \in R(W')$

ii)′ There is an open covering $\{W_i\}$ of W such that for each i there are $W_i \xrightarrow{l_i} \mathbb{Q}$ and $W_i \xrightarrow{r_i} \mathbb{Q}$ with $(W_i \to W, W_i \xrightarrow{l_i} \mathbb{Q}) \in L(W_i)$, and $(W_i \to W, W_i \xrightarrow{r_i} \mathbb{Q}) \in R(W_i)$

iii)′ For any map $\beta : W' \to W$ and any $q, r : W' \to \mathbb{Q}$: if $(\beta, r) \in L(W)$ and $q(x) < r(x)$ for all $x \in W'$, then $(\beta, q) \in L(W')$, and similar for R

iv)′ For any $\beta : W' \to W$ ad $q : W' \to \mathbb{Q}$: if $(\beta, q) \in L(W')$ there is an open covering $\{W'_i\}$ of W', and maps $r_i : W'_i \to \mathbb{Q}$ such that $(\beta \restriction W'_i, r_i) \in L(W'_i)$, and $r_i(x) > q(x)$ for all $x \in W'_i$. And similar for R

v)′ For any $\beta : W' \to W$ and $q, r : W' \to \mathbb{Q}$ satisfying $q(x) < r(x)$ for all $x \in W'$, there is an open covering $\{W'_i\}$ of W' such that for each i, either $(\beta \restriction W'_i, q \restriction W'_i) \in L(W'_i)$ or $(\beta \restriction W'_i, q \restriction W'_i) \in R(W'_i)$.

Let $\hat{q} : W \to \mathbb{Q}$ be the constant function with value q. For every $x \in W$ we define:

$$
\begin{array}{rcl}
L_x & = & \{q \in \mathbb{Q} \,|\, \exists \text{open } V \subseteq W(x \in V \wedge (V \to W, \hat{q} \restriction V) \in L(V))\} \\
R_x & = & \{q \in \mathbb{Q} \,|\, \exists \text{open } V \subseteq W(x \in V \wedge (V \to W, \hat{q} \restriction V) \in R(V))\}
\end{array}
$$

Then you should verify that (L_x, R_x) form a Dedekind cut in Set, hence determine a real number $f_{L,R}(x)$.

By definition of L_x and R_x, if q, r are rational numbers then $q < f_{L,R}(x) < r$ holds if and only if $q \in L_x$ and $r \in R_x$; so the preimage of the open interval (q, r) under $f_{L,R}$ is open; that is, $f_{L,R}$ is continuous. We have therefore defined a map $(L, R) \mapsto f_{L,R} : R_d(W) \to y_{\mathbb{R}}(W)$. It is easy to verify that this gives a map of sheaves: $R_d \to y_{\mathbb{R}}$.

For the other direction, if $f : W \to \mathbb{R}$ is continuous, one defines subsheaves L_f, R_f of $y_{W\times\mathbb{Q}}$ as follows: for $\beta : W' \to W, p : W' \to \mathbb{Q}$ put

$$
\begin{array}{lll}
(\beta, p) \in L_f(W') & \text{iff} & \forall x \in W'(p(x) < f(\beta(x))) \\
(\beta, p) \in R_f(W') & \text{iff} & \forall x \in W'(p(x) > f(\beta(x)))
\end{array}
$$

We leave it to you to verify that then $W \Vdash \text{Cut}(L_f, R_f)$ and that the two given operations between $y_{\mathbb{R}}(W)$ and $R_d(W)$ are inverse to each other. You should observe that every continuous function $f : W \to \mathbb{Q}$ is locally constant, as $\mathbb{Q}$ is discrete. ∎

Corollary 10.28 *The exponential $(R_d)^{R_d}$ is isomorphic to $y_{\mathbb{R}^{\mathbb{R}}}$, where $\mathbb{R}^{\mathbb{R}}$ is the set of continuous maps $\mathbb{R} \to \mathbb{R}$ with the compact-open topology.*

Proof. This follows at once from proposition 10.27, the observation that y preserves exponents, and the fact that $\mathbb{R}$ is locally compact. ∎

From the corollary we see at once that arrows $R_d \to R_d$ in $\mathcal{T}$ correspond bijectively to continuous functions $\mathbb{R} \to \mathbb{R}$, but this is not yet quite Brouwer's statement that all functions (defined, possibly, with extra parameters) from R_d to R_d are continuous. So we prove that now.

Theorem 10.29 $\mathcal{T} \Vdash$ *"All functions $R_d \to R_d$ are continuous"*

Proof. . In other words, we have to prove that the sentence

$$
\begin{array}{l}
\forall f \in (R_d)^{R_d} \forall x \in R_d \forall \epsilon \in R_d(\epsilon > 0 \to \exists \delta \in R_d(\delta > 0 \wedge \\
\quad \forall y \in R_d(x - \delta < y < x + \delta \to f(x) - \epsilon < f(y) < f(x) + \epsilon)))
\end{array}
$$

is true in $\mathcal{T}$.

We can work in $y_{\mathbb{R}^{\mathbb{R}}}$ for $(R_d)^{R_d}$, so $(R_d)^{R_d}(W) = \mathrm{Cts}(W \times \mathbb{R}, \mathbb{R})$. Take $f \in (R_d)^{R_d}(W)$ and $a, \epsilon \in R_d(W)$ such that $W \Vdash \epsilon > 0$. So $f : W \times \mathbb{R} \to \mathbb{R}$, and $a, \epsilon : W \to \mathbb{R}$, $\epsilon(x) > 0$ for all $x \in W$. We have to show:

(∗) $W \Vdash \exists \delta \in R_d(\delta > 0 \wedge \forall y \in R_d(a - y < \delta < a + \delta \to$
$f(a) - \epsilon < f(y) < f(a) + \epsilon))$

Now f and ϵ are continuous, so for each $x \in W$ there is an open neighborhood $W_x \subseteq W$ of x, and a $\delta_x > 0$ such that for each $\xi \in W_x$ and $t \in (a(x) - \delta_x, a(x) + \delta_x)$:

(1) $|a(\xi) - a(x)| < \frac{1}{2}\delta_x$

(2) $|f(\xi, t) - f(\xi, a(x))| < \frac{1}{2}\epsilon(\xi)$

We claim:

$$W_x \Vdash \forall y(a - \frac{1}{2}\delta_x < y < a + \frac{1}{2}\delta_x \to f(a) - \epsilon < f(y) < f(a) + \epsilon)$$

Note that this establishes what we want to prove.

To prove the claim, choose $\beta : V \to W_x$, $b : V \to \mathbb{R}$ such that

$$V \Vdash a\beta - \frac{1}{2}\delta_x < b < a\beta + \frac{1}{2}\delta_x$$

Then for all $\zeta \in V$, $|a\beta(\zeta) - b(\zeta)| < \frac{1}{2}\delta_x$, so by (1),

$$|a(x) - b(\zeta)| < \delta_x$$

Therefore we can substitute $\beta\zeta$ for ξ, and $b(\zeta)$ for t in (2) to obtain

$$|f(\beta(\zeta), b(\zeta)) - f(x, a(x))| < \tfrac{1}{2}\epsilon\beta(\zeta)$$
and
$$|f(\beta(\zeta), a\beta(\zeta)) - f(x, a(x))| < \tfrac{1}{2}\epsilon\beta(\zeta)$$

We conclude that $|f(\beta(\zeta)), b(\zeta)) - f(\beta(\zeta), a\beta(\zeta))| < \epsilon\beta(\zeta)$. Hence,

$$V \Vdash (f\beta)(a\beta) - \epsilon\beta < (f\beta)(b) < (f\beta)(a\beta) + \epsilon\beta$$

which proves the claim and we are done. ∎

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