# Computability Theory 2013
# Solutions of Hand-in Exercises

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**Exercise 21** Let $K : \mathbb{N} \to \mathbb{N}$, $G : \mathbb{N}^{k+1} \to \mathbb{N}$ and $H : \mathbb{N}^{k+3} \to \mathbb{N}$ be functions. Define $F$ by:

$$\begin{array}{rcl} F(0, \vec{y}, x) & = & G(\vec{y}, x) \\ F(z+1, \vec{y}, x) & = & H(z, F(z, \vec{y}, K(x)), \vec{y}, x) \end{array}$$

Suppose that $G$, $H$ and $K$ are primitive recursive.

a) Prove directly, using the pairing function $j$ and suitably adapting the proof of proposition 2.1.9: if $\forall x(K(x) \le x)$, then $F$ is primitive recursive.

b) Define a new function $F'$ by:

$$\begin{array}{rcl} F'(0, m, \vec{y}, x) & = & G(\vec{y}, K^m(x)) \\ F'(n+1, m, \vec{y}, x) & = & H(n, F'(n, m, \vec{y}, x), \vec{y}, K^{m \dot{-} (n+1)}(x)) \end{array}$$

Recall that $K^{m \dot{-} (n+1)}$ means: the function $K$ applied $m \dot{-} (n+1)$ times.

Prove: if $n \le m$ then $\forall k[F'(n, m+k, \vec{y}, x) = F'(n, m, \vec{y}, K^k(x))]$

c) Prove by induction: $F(z, \vec{y}, x) = F'(z, z, \vec{y}, x)$ and conclude that $F$ is primitive recursive, also without the assumption that $K(x) \le x$.

**Solution**: There is more than one way to solve a), which was the most challenging part of the exercise. Define the function $\check{F}$ by:

$$\check{F}(z, \vec{y}, x) \; = \; \langle F(z, \vec{y}, 0), \ldots, F(z, \vec{y}, x)\rangle$$

Then $F(z, \vec{y}, x) = (\check{F}(z, \vec{y}, x))_x$, so if we can show that $\check{F}$ is primitive recursive, then so is $F$, being defined from $\check{F}$ by composition with primitive recursive functions.
Define an auxiliary function $L$ by

$$L(z, u, \vec{y}, x) \; = \; \langle H(z, (u)_{K(0)}, \vec{y}, 0), \ldots, H(z, (u)_{K(x)}, \vec{y}, x)\rangle$$

Then

$$\begin{array}{rcl} L(z, u, \vec{y}, 0) & = & \langle H(z, (u)_{K(0)}, 0)\rangle \\ L(z, u, \vec{y}, x+1) & = & L(z, u, \vec{y}, x) * \langle H(z, (u)_{K(x+1)}, \vec{y}, x+1)\rangle \end{array}$$

so $L$ is defined by primitive recursion from primitive recursive functions, hence primitive recursive.
Now for $\check{F}$ we have:

$$\begin{array}{rcl} \check{F}(0, \vec{y}, x) & = & \langle G(\vec{y}, 0), \ldots, G(\vec{y}, x)\rangle \\ \check{F}(z+1, \vec{y}, x) & = & L(z, \check{F}(z, \vec{y}, x), \vec{y}, x) \end{array}$$

(this takes a few lines of checking!) where in the first line we have a function defined by course-of-values recursion from $G$ (so primitive recursive); and $\check{F}$ is defined by primitive recursion; so it is primitive recursive.

b) The only point here is to get the induction right. If one wishes to show $\forall n \leq m\, P(m)$ then it suffices to show: $P(0)$ and for all $n < m$, if $P(n)$ then $P(n+1)$.

For $n = 0$ we have $F'(n, m+k, \vec{y}, x) = F'(0, m+k, \vec{y}, x) = G(\vec{y}, K^{m+k}(x))$ and also

$$F'(n, m, \vec{y}, K^k(x)) = F'(0, m, \vec{y}, K^k(x)) = G(\vec{y}, K^m(K^k(x))) = G(\vec{y}, K^{m+k}(x)$$

so the statement holds for $n = 0$. Suppose $n < m$ and the statement holds for $n$. Since $n < m$ hence $n+1 \leq m$, we have $m + k \dot{-} (n+1) = (m \dot{-} (n+1)) + k$ (this is the point where the assumption $n < m$ is used! This does not hold in general!), so using the induction hypothesis we have: $F'(n+1, m+k, \vec{y}, x) = H(n, F'(n, m+k, \vec{y}, x), \vec{y}, K^{m+k \dot{-} (n+1)}(x)) = H(n, F'(n, m, \vec{y}, K^k(x)), \vec{y}, K^{m \dot{-} (n+1)}(K^k(x))) = F'(n+1, m, \vec{y}, K^k(x))$. This completes the induction step.

c) We have $F(0, \vec{y}, x) = G(\vec{y}, x)$ and $F'(0, 0, \vec{y}, x) = G(\vec{y}, K^0(x)) = G(\vec{y}, x)$, so for $z = 0$ the statement holds.

Suppose the statement holds for $z$. Since $z + 1 \dot{-} (z+1) = 0$ we have: $F'(z+1, z+1, \vec{y}, x) = H(z, F'(z, z+1, \vec{y}, x), \vec{y}, x) = H(z, F'(z, z, \vec{y}, K(x)), \vec{y}, x)) = H(z, F(z, \vec{y}, K(x)), \vec{y}, x) = F(z+1, \vec{y}, x)$, which completes the induction step.
We see that the function $F$ is defined by composition from $F'$ (and projection functions); hence $F$ is primitive recursive. Since we have never used that $K(x) \leq x$ in this proof, $F$ is primitive recursive without this assumption.

**Exercise 35**.Prove *Smullyan's Simultaneous Recursion Theorem*: given two binary partial recursive functions $F$ and $G$, for every $k$ there exist indices $a$ and $b$ satisfying for all $x_1, \ldots, x_k$:

$$a \cdot (x_1, \ldots, x_k) \simeq F(a, b) \cdot (x_1, \ldots, x_k)$$

and

$$b \cdot (x_1, \ldots, x_k) \simeq G(a, b) \cdot (x_1, \ldots, x_k)$$

**Solution**: First, use the Recursion Theorem to find an index $\alpha$ such that for all $y, x_1, \ldots, x_k$:

$$\alpha \cdot (y, x_1, \ldots, x_k) \;\simeq\; F(S^1_k(\alpha, y), y) \cdot (x_1, \ldots, x_k)$$

Then, again applying the Recursion Theorem, find index $\beta$ such that for all $x_1, \ldots, x_k$:

$$\beta \cdot (x_1, \ldots, x_k) \;\simeq\; G(S^1_k(\alpha, \beta), \beta) \cdot (x_1, \ldots, x_k)$$

Let $b = \beta$ and $a = S^1_k(\alpha, \beta)$. Then:

$$\begin{array}{rcl} a \cdot (\vec{x}) & \simeq & S^1_k(\alpha, \beta) \cdot (\vec{x}) \\ & \simeq & \alpha \cdot (\beta, \vec{x}) \\ & \simeq & F(S^1_k(\alpha, \beta), \beta) \cdot (\vec{x}) \\ & \simeq & F(a, b) \cdot (\vec{x}) \end{array}$$

and

$$\begin{array}{rcl} b \cdot (\vec{x}) & \simeq & \beta \cdot (\vec{x}) \\ & \simeq & G(S^1_k(\alpha, \beta), \beta) \cdot (\vec{x}) \\ & \simeq & G(a, b) \cdot (\vec{x}) \end{array}$$

**Exercise 55**: Conclude from Theorem 3.3.3 that there cannot exist a total recursive function $F$ which is such that for all $e$: $\phi_e$ is constant on its domain if and only if $F(e) \in \mathcal{K}$.

**Solution**: Suppose there were such $F$. Then we have that

$$X \;=\; \{e \mid \phi_e \text{ is constant on its domain}\}$$

is reducible to $\mathcal{K}$ via $F$, so $X$ would be r.e. by Exercise 43.

It is also clear from the definition that $X$ is extensional for indices of partial recursive functions.

Therefore, by Myhill-Shepherdson (3.3.3. part 1)), the set $F = \{\phi_e \,|\, e \in X\}$ is open in $\mathcal{PR}$.

However, this would mean (by the remarks following Exercise 53) that $F$ is upwards closed. Since $F$ contains the empty function, therefore $F$ would be the set of all partial recursive functions; so every partial recursive function would be constant on its domain. This is clearly false.

**Exercise 72**: Find for each of the following relations an $n$, as small as you can, such that they are in $\Sigma_n$, $\Pi_n$ or $\Delta_n$:

i) $\{e \mid W_e \text{ is finite}\}$

ii) $\{e \mid \mathrm{rge}(\phi_e) \text{ is infinite}\}$

iii) $\{e \mid \phi_e \text{ is constant (possibly partial)}\} = \{e \mid \phi_e \text{ has at most one value}\}$

iv) $\{j(e,f) \mid W_e \leq_m W_f\}$

v) $\{e \mid W_e \text{ is } m\text{-complete in } \Sigma_1\}$

Then, classify the first three of these *completely*, by showing that they are $m$-complete in the class you found.

**Solution**: we do i) and ii) simultaneously. Let DomFin be the set $\{e \mid W_e \text{ is finite}\}$ and let RgeInf be the set $\{e \mid \mathrm{rge}(\phi_e) \text{ is infinite}\}$. We have:

$$\begin{array}{rcl} e \in \mathrm{DomFin} & \Leftrightarrow & \exists x \forall y \forall k (T(1,e,y,k) \to y \leq x) \\ e \in \mathrm{RgeInf} & \Leftrightarrow & \forall x \exists y \exists k (T(1,e,y,k) \wedge U(k) > x) \end{array}$$

From this we see that DomFin is in $\Sigma_2$ and RgeInf is in $\Pi_2$.

From the Kleene Normal Form Theorem we know that the set Tot$= \{e \,|\, \forall x \exists y T(1,e,x,y)\}$ is $m$-complete in $\Pi_2$ and its complement NTot $= \mathbb{N} - $ Tot is therefore $m$-complete in $\Sigma_2$. Let $g$ be an index such that

$$g{\cdot}(e,x) \;\simeq\; \left\{ \begin{array}{rl} x & \text{if } \exists z \forall i < x T(1,e,i,(z)_i) \\ \text{undefined} & \text{otherwise} \end{array} \right.$$

Let $G(e) = S^1_1(g,e)$. We have: $\mathrm{rge}(\phi_{G(e)})$ is infinite if and only if $W_{G(e)}$ is infinite, if and only if $e \in$ Tot; so $G$ reduces Tot to RgeInf and NTot to DomFin. Therefore, RgeInf is $m$-complete in $\Pi_2$ and DomFin is $m$-complete in $\Sigma_2$.

iii): let Const be the set from iii). We have

$$e \in \mathrm{Const} \;\Leftrightarrow\; \forall uykl (T(1,e,u,k) \wedge T(1,e,y,l) \to U(k) = U(l))$$

which establishes that Const is in $\Pi_1$.
Let $g$ be an index satisfying:

$$g{\cdot}(e,x) \;\simeq\; \left\{ \begin{array}{rl} 0 & \text{if } \forall y \leq x \neg T(1,e,e,y) \\ z+1 & \text{if } z \leq x \text{ is minimal with } T(1,e,e,z) \end{array} \right.$$

Let $G(e) = S^1_1(g,e)$. We see that $G(e) \in$ Const precisely when $e \in \mathbb{N} - \mathcal{K}$. Since $\mathcal{K}$ is $m$-complete in $\Sigma_1$ hence $\mathbb{N} - \mathcal{K}$ is $m$-complete in $\Pi_1$, we see that Const is $m$-complete in $\Pi_1$.

iv): $W_e \leq_m W_f$ if and only if there is a total recursive function $\phi_u$ such that $W_e = \phi_u^{-1}(W_f)$. Therefore $W_e \leq_m W_f$ holds, if and only if the following condition is satisfied:

$$\begin{array}{ll} \exists u & [\forall x \exists y T(1,u,x,y) \\ & \wedge \\ & \forall zvw \exists a (T(1,e,z,v) \wedge T(1,u,z,w) \to T(1,f,U(w),a)) \\ & \wedge \\ & \forall bcd \exists g (T(1,u,b,c) \wedge T(1,f,U(c),d) \to T(1,e,b,g)] \end{array}$$

We have an existential quantifier before an intersection of $\Pi_2$-sets. Since $\Pi_2$ is closed under intersections (proposition 4.2.4), the set $\{j(e,f) \mid W_e \leq_m W_f\}$ is in $\Sigma_3$.

v): $W_e$ is $m$-complete in $\Sigma_1$ if and only if $\mathcal{K} \leq_m W_e$. So the set of v) is in $\Sigma_3$ by the result of iv).

**Exercise 77**. Prove that for a set $X \subseteq \mathbb{N}$ the following assertions are equivalent:

i) $X$ is creative

ii) $X$ is 1-complete in $\Sigma_1$;

iii) $X$ is $m$-complete in $\Sigma_1$;

iv) There is a total recursive bijective function $h$ such that $h[X] = \mathcal{K}$

Hint: use Exercises 75-76, proposition 4.3.5 and Theorem 4.3.3.

**Solution**: it is necessary to prove first that $\mathcal{K}$ is 1-complete in $\Sigma_1$. In fact the usual proof of $m$-completeness of $\mathcal{K}$ works, because $Smn$-functions can be assumed to be injective.
i)$\Rightarrow$ii): Suppose $X$ is creative. Then by 4.3.5, $\mathcal{K} \leq_1 X$. Since $\mathcal{K}$ is 1-complete, $X$ is.
ii)$\Rightarrow$iii): trivial.
iii)$\Rightarrow$iv): Suppose $X$ is $m$-complete in $\Sigma_1$. Then $\mathcal{K} \leq_m X$. Since $\mathcal{K}$ is creative by Exercise 75, $X$ is creative by Exercise 76iii); so $\mathcal{K} \leq_1 X$. Because $\mathcal{K}$ is 1-complete we also have $X \leq_1 \mathcal{K}$. Statement iv) now follows from Theorem 4.3.3.
iv)$\Rightarrow$i): Suppose $h : \mathbb{N} \to \mathbb{N}$ is a total recursive bijection with $h[X] = \mathcal{K}$. Let $G$ be primitive recursive such that $W_{G(e)} = h[W_e]$ for all $e$. By 4.3.4, we may assume that $\mathcal{K}$ is creative via a total recursive, injective function $H$. Let $F(e) = h^{-1}(H(G(e)))$. We claim that $X$ is creative via $F$. Indeed, suppose $W_e \cap X = \emptyset$. Then $W_{G(e)} \cap \mathcal{K} = \emptyset$. So $H(G(e)) \notin W_{G(e)} \cup \mathcal{K}$. Then $F(e) = h^{-1}(H(G(e))) \notin W_e \cup X$.

**Exercise 87**. Given sets $A$ and $B$, prove that the following assertions are equivalent:

i) $B \leq_T A$

ii) There exist total recursive functions $F$ and $G$ such that the following holds:

$$\begin{array}{lll} x \in B & \text{if and only if} & \exists\sigma(\sigma \in W_{F(x)} \wedge \forall i < \mathrm{lh}(\sigma)(\sigma)_i = \chi_A(i)) \\ x \notin B & \text{if and only if} & \exists\sigma(\sigma \in W_{G(x)} \wedge \forall i < \mathrm{lh}(\sigma)(\sigma)_i = \chi_A(i)) \end{array}$$

(Hint: use proposition 5.1.8.)

**Solution**: i)$\Rightarrow$ii): suppose i) holds. By proposition 5.1.8 we know that there is a number $e$ such that for all $x$:

$$\begin{array}{lll} x \in B & \text{if and only if} & \exists\sigma(\sigma \preceq \chi_A \wedge \exists w(T^\sigma(1,e,x,w) \wedge U(w) = 0)) \\ x \notin B & \text{if and only if} & \exists\sigma(\sigma \preceq \chi_A \wedge \exists w(T^\sigma(1,e,x,w) \wedge U(w) = 1)) \end{array}$$

where we use $\sigma \preceq \chi_A$ as short for: $\forall i < \mathrm{lh}(\sigma)\,(\sigma)_i = \chi_A(i)$. Let $f$ and $g$ be indices such that

$$f\cdot(x,y) \simeq \begin{cases} 0 & \text{if } \exists w(T^y(1,e,x,w) \wedge U(w) = 0) \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$g\cdot(x,y) \simeq \begin{cases} 0 & \text{if } \exists w(T^y(1,e,x,w) \wedge U(w) = 1) \\ \text{undefined} & \text{otherwise} \end{cases}$$

and put $F(x) = S^1_1(g,x)$, $G(x) = S^1_1(g,x)$. Then

$$\begin{array}{lll} W_{F(x)} & = & \{\sigma \mid \exists w(T^\sigma(1,e,x,w) \wedge U(w) = 0)\} \\ W_{G(x)} & = & \{\sigma \mid \exists w(T^\sigma(1,e,x,w) \wedge U(w) = 1)\} \end{array}$$

Then ii) holds: suppose $x \in B$. Then by the choice of $e$ we have $\exists\sigma(\sigma \preceq \chi_A \wedge \exists w(T^\sigma(1, e, x, w) \wedge U(w) = 0))$ so $\exists\sigma(\sigma \preceq \chi_A \wedge \sigma \in W_{F(x)})$. The converse is immediate; and a similar equivalence holds for $x \notin B$.

ii)⇒i): suppose ii) holds. In order to determine $\chi_B(x)$, find the least pair $\langle\sigma, w\rangle$ satisfying $\sigma \preceq \chi_A$ and $w$ testifies that $\sigma \in W_{F(x)}$ or $\sigma \in W_{G(x)}$. Note that only one of the two can happen. Output 0 if $\sigma \in W_{F(x)}$ and 1 if $\sigma \in W_{G(x)}$. This is recursive in $A$, so $B \leq_T A$.