Seminar Hilbert 10 - Homework 5

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Due: October 28

In these exercises, $\mathcal{F}_0$ is the class of functions in real variables represented by expressions combining the variables with integers and the number π by addition, substraction, multiplication and the sin function.

Exercise 1 We have shown that there is no algorithm for deciding for an arbitrary $\Phi(\chi_1, \ldots, \chi_m) \in \mathcal{F}_0$ whether

$$\exists \chi_1, \ldots, \chi_m \Phi(\chi_1, \ldots, \chi_m) = 0$$

In this exercise, we'll improve this result by showing that it is undecidable to determine whether

$$\exists \chi_1, \ldots, \chi_m \Phi(\chi_1, \ldots, \chi_m) < 1$$

for arbitrary $\Phi \in \mathcal{F}_0$. To do this, we need to modify the function

$$D^2(\chi_1, \ldots, \chi_m) + \sin^2(\pi\chi_1) + \ldots + \sin^2(\pi\chi_m)$$

such that it takes values 0 precisely when the above does, and values > 1 otherwise. Of course, the function still needs to be in $\mathcal{F}_0$.

a) For $\chi_1, \ldots, \chi_m$ arbitrary real numbers, we denote by $(x_1, \ldots, x_m)$ the point with integer coordinates closest to $(\chi_1, \ldots, \chi_m)$. The distance between them is denoted by ϵ. Show that there is a polynomial B (computable in D!) such that for all $\chi_1, \ldots, \chi_m$:

$$|D^2(\chi_1, \ldots, \chi_m) - D^2(x_1, \ldots, x_m)| < B(\chi_1, \ldots, \chi_m)\epsilon.$$

Hint: Use Taylor's theorem.

Conclude that for

$$\epsilon < \frac{1}{2B(\chi_1, \ldots, \chi_m)} \tag{1}$$

we have

$$D^2(\chi_1, \ldots, \chi_m) > \frac{1}{2}$$

if $D(x_1, \ldots, x_n) \neq 0$.

b) Show that if (1) does not hold, we have:

$$\sin^2(\pi\chi_1) + \ldots + \sin^2(\pi\chi_m) \geq \frac{1}{B^2(\chi_1, \ldots, \chi_m)}.$$

c) Conclude that there is no algorithm for deciding for an arbitrary function $\Phi \in \mathcal{F}_0$ whether the inequality

$$\Phi(\chi_1, \ldots, \chi_m) < 1.$$

has a solution in real $\chi_1, \ldots, \chi_m$.

Exercise 2 In this exercise, we'll improve the undecidability result about functions in $\mathcal{F}_0$ to the same result about functions in $\mathcal{F}_0$ with only one real variable. The key is to prove that the image of the map

$$\chi \mapsto (\chi \sin(\chi), \chi \sin(\chi^3), \ldots, \chi \sin(\chi^{2m-1}))$$

lies dense in $\mathbb{R}^m$. For every m, we denote this map by f_m.

We will first prove the case where $m = 2$.

a) Let $y_1, y_2, \delta \in \mathbb{R}$ be arbitary real numbers, with $\delta > 0$. Show that there are reals χ_1, χ_2 such that:

(i) $\chi_2 > \chi_1 > |y_2|$

(ii) $\chi_2 \sin(\chi_2) = y_1$

(iii) $\chi_2^3 - \chi_1^3 > 2\pi$

(iv) $(\chi_2 - \chi_1)(\chi_2 + 1) < \delta$.

Hint: Choose an appropriate χ_1 and define $\chi_2 = (\chi_1^2 + \delta/2)^{1/2}$.

b) Let $y_1, y_2, \delta \in \mathbb{R}$ be arbitary real numbers, with $\delta > 0$. Show that there is a χ such that

$$|f_1(\chi) - y_1| < \delta \text{ and } f_2(\chi) = y_2.$$

Conclude that the image of f_2 lies dense in $\mathbb{R}^2$. *Hint:* Choose an appropriate χ between χ_1 and χ_2 in (a) and use the mean value theorem on $f_1(\chi) - y_1 = f_1(\chi) - f_1(\chi_2)$ to make an estimate.

c) Let $y_1, \ldots, y_m, y_{m+1}, \delta \in \mathbb{R}$ be arbitary real numbers, with $\delta > 0$. Suppose f is a function such that $f(]\chi, \infty[)$ lies dense in $\mathbb{R}^m$ for any χ. Show that there are reals χ_1, χ_2 such that:

(i) $\chi_2 > \chi_1 > |y_{m+1}|$

(ii) $|f(\chi_2) - (y_1, \ldots, y_m)| < \delta$

(iii) $\chi_2^{2m+1} - \chi_1^{2m+1} > 2\pi$

(iv) $(\chi_2 - \chi_1)((2m-1)\chi_2^{2m-1} + 1) < \delta$.

Hint: Modify the proof in (a).

d) Prove that the image of f_m lies dense in $\mathbb{R}^m$, for every $m \geq 1$.

e) Show, using exercise 1, that there is no algorithm for deciding for an arbitary function $\Psi \in \mathcal{F}_0$ in one real variable whether the equation

$$\Psi(\chi) < 1$$

has a real solution χ.