Homework set 6

Hilbert's tenth problem seminar, Fall 2013, due November 4th

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Exercise 1: Over all integers (not only positive ones), we can define a new successor operation S (Where $S(a)$ gives the smallest number higher than a) . The first three exercises are about arithmetically defining things in terms of this S and multiplication $(\cdot)$.

a) Arithmetically define 0 (zero) in terms of S and multiplication $(\cdot)$ over all integers.
b) Use the result in a) to prove that any constant $n\epsilon\mathbf{Z}$ is arithmetically definable in terms of S and multiplication $(\cdot)$.
c) Show that addition of integers is arithmetically definable in terms of the successor operator S and multiplication $(\cdot)$. (Hint: Improve the original result which used the statement $S(a \cdot c) \cdot S(b \cdot c) = S[(c \cdot c) \cdot S(a \cdot b)]$)

Exercise 2: Over the rationals, we have seen that the notion of an integer (Int) is arithmetically definable in terms of + and $\cdot$ over rationals. Use this to prove that the following concepts are arithmetically definable in terms of addition (+) and multiplication $(\cdot)$.

a) Prove that $a = den(b)$ is arithmetically definable in terms of + and $\cdot$ over rationals. Where $den(b)$ is the smallest possible positive denominator of b. (example: den(-6/8) = 4)
b) Prove that $a > b$ is arithmetically definable in terms of + and $\cdot$ over rationals.
c) Prove that $a = \lfloor b \rfloor$ is arithmetically definable in terms of + and $\cdot$ over rationals. Where $\lfloor b \rfloor$ is b rounded down to an integer (floor operator).

Exercise 3: Consider the following system of axioms using the symbol Pos (meaning it is a positive integer) and the two mathematical constants + and $\cdot$:
B1: $\exists c : \mathcal{U}(c)$
B2: $\forall a, b, c : \{[Pos(a) \wedge Pos(b) \wedge \mathcal{U}(c) \wedge (a + c = b + c)] \rightarrow a = b\}$
B3: $\forall a, b, c : \{[Pos(a) \wedge Pos(b) \wedge \mathcal{U}(c)] \rightarrow a + (b + c) = (a + b) + c\}$
B4: $\forall a, c : \{[Pos(a) \wedge \mathcal{U}(c)] \rightarrow a \cdot c = a\}$
B5: $\forall a, b, c : \{[Pos(a) \wedge Pos(b) \wedge \mathcal{U}(c)] \rightarrow a \cdot (b + c) = (a \cdot b) + a\}$
B6: $\forall c : \{[\mathcal{U}(c) \wedge \Phi(c) \wedge (\forall a : [Pos(a) \wedge \Phi(a)] \rightarrow \Phi(a + c))] \rightarrow \forall a : (Pos(a) \rightarrow \Phi(a))\}$

Where $\mathcal{U}(c)$ is true if and only if $\{Pos(c) \wedge (\forall x, y : [Pos(x) \wedge Pos(y)] \rightarrow\sim (x + y = c))\}$, hence $\mathcal{U}(c) \leftrightarrow c = 1$.
In this axiom system, B6 is true for any statement $\Phi(a)$ with free variable a. B6 basically says that we can use induction on one positive integer.

Prove that the following 3 statements (B7,B8,B9) are provable in terms of the axioms in the axiom system defined above (B1-B6):
B7: $\forall a, b, c : \{[Pos(a) \wedge Pos(b) \wedge Pos(c) \wedge (a + c = b + c)] \rightarrow a = b\}$
B8: $\forall a, b : \{[Pos(a) \wedge Pos(b)] \rightarrow a + b = b + a\}$
B9: $\forall a, b, c : \{[Pos(a) \wedge Pos(b) \wedge Pos(c)] \rightarrow a + (b + c) = (a + b) + c\}$