Hilbert's Tenth Problem Seminar
Homework set 7

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(Due to Nov. 18)

Exercise 1. Let $\mathbb{A}(d)$ be any quadratic ring and let

$$\omega = \begin{cases} \sqrt{d} & \text{if } d \equiv 2,3 \mod 4 \\ \frac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \mod 4 \end{cases}.$$

Prove that for every element $x \in \mathbb{A}(d)$ there are $a, b \in \mathbb{Z}$ such $x = a + b\omega$.

Exercise 2. Let $\mathbb{Q}(\sqrt{d})$ is a quadratic number field.

(a) Show that the norm is multiplicative, i.e., if $x, y \in \mathbb{Q}(\sqrt{d})$ then we have $N(xy) = N(x)N(y)$.

(b) Show that if $n \in \mathbb{N}$ and $x \in \mathbb{Q}(\sqrt{d})$ then $N(nx) = n^2 N(x)$.

(c) Show that if $d \leq 1$ then $N(x) \geq 0$ for any $x \in \mathbb{Q}(\sqrt{d})$.

(d) Show that if $x \in \mathbb{Q}(\sqrt{d})$ is a unit, then $N(x) = \pm 1$.

Exercise 3. Let n, k and a be natural numbers with $a > 1$. Show that the integral solutions to Pell's equation can be computed recursively by

$$x_{nk}(a) + y_{nk}(a)\sqrt{a^2-1} = \left(x_n(a) + y_n(a)\sqrt{a^2-1}\right)^k.$$

Conclude that, writing $x_s = x_s(a)$ and $y_s = y_s(a)$, that

$$y_{nk} = \sum_{\substack{i=1 \\ i \text{ odd}}} \binom{k}{i} (x_n)^{k-i} (y_n)^i (a^2-1)^{(i-1)/2}.$$

Exercise 4. Let $\mathbb{A}(d)$ be any quadratic ring and let $y \in \mathbb{A}(d)$. Show that if $y^2 \in \mathbb{Q}$, then $y^2 \in \mathbb{Z}$. Furthermore, show that if $d > 1$, $y^2 \in \mathbb{N}$.

Exercise 5. Let $\mathbb{A}(d)$ be any imaginary quadratic ring.

(a) Show that the only possible units are

$$\pm 1, \pm i, \frac{\pm 1 \pm i\sqrt{3}}{2}.$$

(b) Use this to prove that the fact that $5h + 2$ is a unit, for $h \in \mathbb{A}(d)$, is contradictory.