Seminar on Hilbert's 10th Problem

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During the seminar, we used the following result by Jan Denef to prove our main theorem:

Let $n > 1$ be a fixed integer. Define $x|^n y$ by $\exists q, f \in \mathbb{Z} : yn^f = xq$. Then the existential problem of $(\mathbb{Z}; +, |^n)$ is undecidable.

In the proof, the Diophantine problem for integers (which we know is undecidable) is coded into this problem. The most difficult part of this proof is showing that $u = z^2$ can be coded into this problem. These two exercises do not solve this, but show two of the most important steps in the proof, hopefully giving you an idea about the entire proof.

Exercise 1

Let $n > 1$, let $u, x, z \in \mathbb{Z}$ and suppose the following conditions hold:

$$nz + nx - 1|^n n^2u - (nx - 1)^2$$

$$2nz + 1|^n nx - 1$$

$$2nz - 1|^n nx - 1$$

$$2n^2u + 1|^n nx - 1$$

We want to prove that $u = z^2$.

a) Using the first condition, prove that $nz + nx - 1|n^2u - n^2z^2$.

Assume $u \neq z^2$
b) Prove that $|nx - 1| - n|z| \leq n^2|u| + n^2z^2$.

c) Using the second and third condition, prove that $(2nz + 1)(2nz - 1)|nx - 1$ and therefore that $4n^2z^2 - 1 \leq |nx - 1|$.

d) Using the fourth condition, prove that $2n^2|u| - 1 \leq |nx - 1|$ and combining this with b) and c), show that $(n|z|)^2 - n|z| - 1 \leq 0$.

e) Conclude that $u = z^2$ must hold.

Exercise 2

Prove that for any integer $d \neq 0$, there exists an integer x such that $x|^n 1$ and $d|^n nx - 1$.
Hint: Split d into two parts and consider the Euler-Phi function on one of these parts.