Seminar on Hilbert's Tenth Problem
Homework, due October 14 - model solution

1a) For the first part, we use induction on n.
Basis. For $n = 0$, we have $\alpha_b(0) = 0 = -\alpha_b(0)$. For $n = 1$, we have $\alpha_b(1) = b\alpha_b(0) - \alpha_b(-1) = -\alpha_b(-1)$, so $\alpha_b(-1) = -\alpha_b(1)$.
Step. Suppose $\alpha_b(-n) = -\alpha_b(n)$ and $\alpha_b(-(n+1)) = -\alpha_b(n+1)$ for some $n \in \mathbb{N}$. We get

$$\alpha_n(-(n+2)) = b\alpha_b(-(n+1)) - \alpha_b(-n) = -(b\alpha_b(n+1) - \alpha_b(n)) = -\alpha_b(n+2).$$

This completes the induction.

For the second part, we observe that

$$\begin{aligned} A_b(n)B_b &= \begin{pmatrix} b\alpha_b(n+1) - \alpha_b(n) & -\alpha_b(n+1) \\ b\alpha_b(n) - \alpha_b(n-1) & -\alpha_b(n) \end{pmatrix} \\ &= \begin{pmatrix} \alpha_b(n+2) & -\alpha_b(n+1) \\ \alpha_b(n+1) & -\alpha_b(n) \end{pmatrix} = A_b(n+1), \end{aligned}$$

for all $n \in \mathbb{Z}$. (Note that this is the same calculation as given during the presentation, only it works for all $n \in \mathbb{Z}$. One may also make this remark and claim the result of the calculation.) Since $A_b(0) = I_2$, we have $A_b(n) = B_b^n$ for all $n \in \mathbb{Z}$. □

1b) We have

$$A_b^{-1}(n) = B_b^{-n} = A_b(-n) = \begin{pmatrix} \alpha_b(-n+1) & -\alpha_b(-n) \\ \alpha_b(-n) & -\alpha_b(-n-1) \end{pmatrix} = \begin{pmatrix} -\alpha_b(n-1) & \alpha_b(n) \\ -\alpha_b(n) & \alpha_b(n+1) \end{pmatrix}.$$

□

1c) We have $\alpha_b(m+1) \equiv \alpha_b(m-1) \mod v$, so

$$A_b(m) \equiv \begin{pmatrix} \alpha_b(m+1) & -\alpha_b(m) \\ \alpha_b(m) & -\alpha_b(m-1) \end{pmatrix} \equiv \begin{pmatrix} \alpha_b(m-1) & -\alpha_b(m) \\ \alpha_b(m) & -\alpha_b(m+1) \end{pmatrix} \equiv -A_b^{-1}(m) \mod v.$$

□

1d) According to exercise (c), we have $A_b^2(m) \equiv -A_b^{-1}(m)A_b(m) \equiv -I_2 \mod v$. This gives

$$\begin{aligned} A_b^n &\equiv B_b^n \equiv B_b^{2lm\pm j} \equiv \left((B_b^m)^2\right)^l \left(B_b^j\right)^{\pm 1} \equiv \left(A_b^2(m)\right)^l (A_b(j))^{\pm 1} \equiv (-I_2)^l (A_b(j))^{\pm 1} \\ &\equiv \pm (A_b(j))^{\pm 1} \mod v. \end{aligned}$$

□

1e) According to exercise (d), we have $A_b(n) \equiv \pm A_b(j) \mod v$ or $A_b(n) \equiv \pm A_b^{-1}(j) \mod v$. That is:

$$\begin{pmatrix} \alpha_b(n+1) & -\alpha_b(n) \\ \alpha_b(n) & -\alpha_b(n-1) \end{pmatrix} \equiv \pm \begin{pmatrix} \alpha_b(j+1) & -\alpha_b(j) \\ \alpha_b(j) & -\alpha_b(j-1) \end{pmatrix} \mod v,$$

or

$$\begin{pmatrix} \alpha_b(n+1) & -\alpha_b(n) \\ \alpha_b(n) & -\alpha_b(n-1) \end{pmatrix} \equiv \pm \begin{pmatrix} -\alpha_b(j-1) & \alpha_b(j) \\ -\alpha_b(j) & \alpha_b(j+1) \end{pmatrix} \mod v.$$

Comparing the bottom left coefficients, we immediately see that, in both cases, $\alpha_b(n) \equiv \pm\alpha_b(j) \mod v$. □

2a) Suppose that $x = \alpha_b(m)$ for some $m \in \mathbb{N}$. Define $y = \alpha_b(m+1)$. Then x and y satisfy the characteristic equation, that is $x^2 - bxy + y^2 = 1$. Now we multiply by 4 and split off the square:

$$\begin{aligned} x^2 - bxy + y^2 = 1 &\Leftrightarrow 4x^2 - 4bxy + 4y^2 = 4 \\ &\Leftrightarrow 4x^2 - (bx)^2 + ((bx)^2 - 4bxy + 4y^2) = 4 \\ &\Leftrightarrow (4 - b^2)x^2 + (2y - bx)^2 = 4 \\ &\Leftrightarrow (2y - bx)^2 = 4 + (b^2 - 4)x^2. \qquad (1) \end{aligned}$$

Since $2y - bx$ clearly is an integer, $4 + (b^2 - 4)x^2$ is a square.

For the other direction, suppose that $4 + (b^2 - 4)x^2$ is a square. We can write $4 + (b^2 - 4)x^2 = k^2$ for a certain $k \in \mathbb{N}$. We have

$$k \equiv k^2 \equiv 4 + (b^2 - 4)x^2 \equiv b^2x^2 \equiv bx \mod 2.$$

This means the number $k + bx$ is even. Since b, x and k are natural numbers, we have $k + bx \geq 0$. So we can write $k + bx = 2y$ for a certain natural number y. We now have $4 + (b^2 - 4)x^2 = k^2 = (2y - bx)^2$. Using equivalence (1) in the other direction, we obtain $x^2 - bxy + y^2 = 1$. So x and y are natural numbers satisfying the characteristic equation, so $x = \alpha_b(m)$ for some $m \in \mathbb{N}$. □
(For $b = 2$ the statement is quite trivial, stating that x is a natural number iff 4 is a square.)

2b) We proceed by induction on n.
Basis. We calculate $F_2 = F_1 + F_0 = 1 + 0 = 1$. For $n = 0$, we have $\alpha_3(0) = 0 = F_0$ and for $n = 1$, we have $\alpha_3(1) = 1 = F_2$.
Step. Suppose that $\alpha_3(n) = F_{2n}$ and $\alpha_3(n+1) = F_{2n+2}$ for some $n \in \mathbb{N}$. We get

$$\begin{aligned} \alpha_3(n+2) &= 3\alpha_3(n+1) - \alpha_3(n) = 3F_{2n+2} - F_{2n} = 2F_{2n+2} + (F_{2n+2} - F_{2n}) = 2F_{2n+2} + F_{2n+1} \\ &= F_{2n+2} + (F_{2n+2} + F_{2n+1}) = F_{2n+2} + F_{2n+3} = F_{2n+4}. \end{aligned}$$

This completes the induction.

According to exercise (a) for $b = 3$, we have that $4 + (3^2 - 4)x^2$ is a square if and only if $x = \alpha_3(n)$ for some $n \in \mathbb{N}$. That is, $5x^2 + 4$ is a square if and only if x is of the form $x = F_{2n}$. □

2c) Note that

$$\begin{aligned}\exists x \in \mathbb{N} \text{ such that } x^2 - (c^2-1)y^2 = 1 &\Leftrightarrow \exists x \in \mathbb{N} \text{ such that } x^2 = 1 + (c^2-1)y^2 \\ &\Leftrightarrow 1 + (c^2-1)y^2 \text{ is a square} \\ &\Leftrightarrow 4(1 + (c^2-1)y^2) \text{ is a square} \\ &\Leftrightarrow 4 + ((2c)^2 - 4)y^2 \text{ is a square} \\ &\Leftrightarrow y = \alpha_{2c}(n) \text{ for some } n \in \mathbb{N}.\end{aligned}$$

This establishes the equality of the two given sets. □

Marking scheme

1a) 2 pt. (1 pt. for each result)
1b) 1 pt.
1c) 2 pt.
1d) 3 pt.
1e) 2 pt.

2a) 5 pt.
I) 2 pt. The equivalence (1), possibly in only one direction.
II) 1 pt. Finishing the proof in the left-to-right-direction.
III) 2 pt. Finishing the proof in the right-to-left-direction. 1 pt. may be given for an important partial result (apart from (1)), such as introducing the number $k + bx$ or considering the equation modulo 2.
2b) 3 pt. (2 pt. for the first part, 1 pt. for the second part)
2c) 2 pt.

Grade = (number of points)/2.