Homework assignment - O-minimal Structures

In exercise 1 and 3 we fix an o-minimal expansion $(R, <, \mathcal{S})$ of an ordered abelian group $(R, <, 0, -, +)$.

Exercise 1 (Exercise 6, page 97) *(4 points)*

Let $X \subseteq R^m$ be definable. Show that there are definable maps $\epsilon : \partial X \to (0, \infty)$ and $\Gamma : A \to X$, where $A = \{(a, t) \in \partial X \times R : t \in (0, \epsilon(a)\}$, such that for each $a \in \partial X$ the function $t \mapsto \Gamma(a, t) : (0, \epsilon(a)) \to X$ is continuous, injective and satisfies $\lim_{t \to 0} \Gamma(a, t) = a$.

Exercise 2 *(2 points)*

Let $f : X \to Y$ be a continuous map from a topological space X into a Hausdorff space Y. Show that its graph $\Gamma(f)$ is a closed subset of $X \times Y$.

Exercise 3 (Exercise 7, page 97) *(4 points)*

Given a map $f : A \to R^n$, $A \subseteq R^m$, we call f locally bounded if each point $a \in A$ has a neighborhood U in A such that $f(U)$ is bounded. Let $A \subseteq R^m$ be definable and $f : A \to R^n$ definable. Prove the following equivalence:

$$f \text{ is continuous} \Leftrightarrow f \text{ is locally bounded and } \Gamma(f) \text{ is closed in } A \times R^n.$$