O-minimal Structures - Assignment 5

14 november 2014

1

Let $(R, <, S)$ be an o-minimal structure. Let A be a cell in R^{m+1}. Note that $\pi(A)$ is a cell in R^m, where $\pi : R^{m+1} \to R^m$ is the projection on the first m coordinates

a) Prove that $\pi^{-1}(\{x\}) \cap A$ is definably connected, for all $x \in \pi(A)$. Hint: use the definition.

b) Assume that A is definably disconnected, i.e. there exists nonempty definably open subsets $U, V \subset X$, such that $A = U \cup V$ and $U \cap V = \emptyset$. Show that $\pi^{-1}(\{x\}) \cap A \subset U$ or $\pi^{-1}(\{x\}) \cap A \subset V$, for all $x \in \pi(A)$.

c) Let $C = \{x \in \pi(A) : \pi^{-1}(\{x\}) \cap A \subset U\}$ and $D = \{x \in \pi(A) : \pi^{-1}(\{x\}) \cap A \subset V\}$. Prove that C and D are open in $\pi(A)$. Hint: there is an obvious open mapping.

d) Prove that if $\pi(A)$ is definably connected, then A is definably connected. And prove by using induction on m, that each cell is definably connected.

2

Let $(R, <, S)$ be an o-minimal structure. Let A be a cell in R^m and Y be a definable set in R^{m+1} and let $p : R^m \to R^k$ be defined as on page 51. Then $p(A)$ is an open cell in R^k. Let $id : R \to R$ be the identity map. Assume that there exists $n \in \mathbb{N}$, such that $|(p, id)Y_{p(x)}| < n$ for all $x \in p(A)$. Prove that there exists $m \in \mathbb{N}$, such that $|Y_x| < m$ for all $x \in A$.