Tame Topology and O-minimal Structures-Dimensions, Homework Set

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In the following exercises we fix an O-minimal structure $(R, <, \mathcal{S})$:

Exercise 1: (**3 points**) (Dimensions of Sets from Definable Families)
Let A and B be definable subsets of R^{m+n}, with A non-empty. Assume that, for every $a \in R^m$, $\dim(B_a) < \dim(A_a)$. Prove that $\dim B < \dim A$. (For definition of A_a and B_a, check p.59 **(3.1)**.)

Exercise 2: (Local Dimension, p.69 **(1.17)** Exercise 2, 3, 4.)

1. (**2 points**) Let $A \subseteq R^m$ be definable and $a \in R^m$. Show there is a number $d \in \{-\infty, 0, \cdots, \dim A\}$ such that there is an open box $U \subseteq R^m$ with $a \in U$, and for all open box $V \subseteq R^m$, if $a \in V$ and $V \subseteq U$, then $\dim(V \cap A) = d$.

   Remark: The number d defined by this property is called the **local dimension of** A **at** a**,** notation $\dim_a(A)$. Note that $\dim_a(A) = -\infty$ iff $a \notin \mathrm{cl}(A)$.

2. (**2 points**) Show that if $A \subseteq R^m$ is a d-dimensional cell, then $\dim_a(A) = d$ for all $a \in \mathrm{cl}(A)$. (**Hint:** use the homeomorphism p_A defined in p.51 **(2.7)**.)

3. (**3 points**) Let $A \subseteq R^m$ be a definable set and $d \in \{0, \cdots, \dim A\}$. Show that the set $\{a \in R^m : \dim_a(A) \geq d\}$ is a definable closed subset of $\mathrm{cl}(A)$. (**Hint:** apply cell decomposition theorem to $\mathrm{cl}(A)$, then show the set $\{a \in R^m : \dim_a(A) \geq d\}$ is the closure of a finite union of cells.)

   Show also that if $A \neq \emptyset$, then $\dim(\{a \in \mathrm{cl}(A) : \dim_a(A) < d\}) < d$.