Tame Topology Seminar - Homework 4

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Exercise 1(3 points. *Let $A \in R^2$ be definable such that A_x is finite for each $x \in R$. Show that there are points $a_1 < \cdots < a_k$ such that the intersection of A with each vertical strip $(a_i, a_{i+1}) \times R$ has the form $\Gamma(f_{i,1}) \cup \cdots \cup \Gamma(f_{i,n(i)}$ for certain definable continuous functions $f_{i,j} : (a_i, a_{i+1}) \to R$ with $f_{i,1}(x) < \cdots < f_{i,n(i)}(x)$ for each $x \in (a_i, a_{i+1})$, we have set $a_0 = -\infty$ and $a_{k+1} = +\infty$. (Hint: use the functions defined in the proof of the finiteless lemma and then apply the monotonicity theorem)*

The finiteness lemma gives us a number $N \in \mathbb{N}$ such that $|A_x| < N$ for all N (0.5 point). We can now define functions $f_1, \cdots f_N$ as in the proof of the finiteness lemma (0.5 point).

$$f_i : \{x \in R : |A_x| \geq i\}, \qquad x \mapsto \text{ ith element of } A_x.$$

We notice that the domain of f_i is definable for every i and the functions f_i are definable as well. This means that we can write the domain of a function f_i as the finite union of intervals and points $\bigcup_{j=1}^{n(i)} I_{i,j} \cup \{x_i, \cdots, x_{n(i)}$ (0.5 point). Restricting f_i to one of these sub-intervals $I_{i,j}$ we find a decomposition of $I_{i,j}$ into intervals by the monotonicity theorem such that f is continuous on each of these sub-intervals (0.5 point). We shall call these new intervals $I_{i,j}$ again, by abuse of notation. Doing this for every i, we obtain a big (but finite) number of intervals. Now using the fact that the intersection of an interval is either empty or a new interval, we take all possible intersections of all these intervals. To be precise we consider the collection (where we index the intervals by 1 up to m)

$$\mathcal{B} = \{\bigcup_{r=1}^{k} \bigcap_{\sigma \in S_k} I_{\sigma r} \subset A : 1 \leq k \leq n\},$$

where we consider all possible intersections of k different intervals (using permutation notation). If we again numerate these intervals $I_1, \cdots I_n$ then on every interval, the function f_i is either continuous or not defined, so we can write $(I_j \times R) \cap A = \Gamma(f_i) \cup \cdots \cup \Gamma(f_{t(j)})$, where $t(j)$ is maximal such that $f_{t(j)}$ is defined on I_j.

Now note that the collection $\mathcal{B}$ still covers A so we are done. (1 point for the last part of the argument, there are many ways to do this but half a point will be subtracted if no refinement argument is considered).

We will now use the previous exercise to show that if A has infinite fibers, its boundary consists of graphs of continuous definable functions.

Exercise 2 (1 point) *Let $A \in R^n$ be definable such that A_x is infinite for each $x \in R$. Show that there are points $a_1 < \cdots < a_k$ such that the intersection of $B_{d2}(A) := \{(x, r) \in A : r \in \mathrm{bd}(A_x)\}$ with each vertical strip $(a_i, a_{i+1}) \times R$ has the form $\Gamma(f_{i,1}) \cup \cdots \cup \Gamma(f_{i,n(i)}$ for certain definable continuous functions*

$f_{i,j} : (a_i, a_{i+1}) \to R$ with $f_{i,1}(x) < \cdots < f_{i,n(i)}(x)$ for each $x \in (a_i, a_{i+1})$, we have set $a_0 = -\infty$ and $a_{k+1} = +\infty$.

It is a result of chapter 1 that the boundary of a definable set is finite (0.5 point), this implies that $B_{d2}(A)$ has finite fibers and so we can apply exercise (1) to obain the desired result (0.5 point).

Exercise 3 (2 points) *Let $f : [a, b] \to R$ be continuous and definable. Show that f takes a maximum and a minimum value on $[a, b]$.*

The monotonicity theorem gives us points $a, a_1, \cdots, a_k, b$ such that f is constant or strictly monotone on the subintervals (0.5 point).

We know that on every sub-interval (a_i, a_{i+1}), the function f is either constant or strictly monotone, which means that the maximum and minimum value it takes on $[a_i, a_{i+1}]$, it must take in the endpoints (1 point).

Globally this means that $\max_{x\in[a,b]} f(x) = \max\{f(a), f(a_1), \cdots, f(a_k), f(b)\}$. and the same for the minimum, therefore the maximum/minimum exists (0.5 point).

Exercise 4 (2 points) *Let I and J be intervals and $f : I \to R$ and $g : J \to R$ strictly monotone definable functions such that $f(I) \subset g(J)$ and $lim_{x\to r(I)} f(x) = \lim_{x\to r(J)} g(t)$ in R_∞, where $r(I)$ and $r(J)$ are the right endpoints of the intervals I and J in R_∞. Show that near these right endpoints f and g are reparametrisations of each other, that is there are subintervals I' of I and J with $r(I) = r(I'), r(J) = r(J')$ and a strictly increasing definable bijection $h : I' \to J'$ such that $f(x) = g(h(x))$ for all $s \in I'$.*

Note that since $f(I) \subset G(J)$ and since their right limits agree, either f, g are both increasing or both decreasing.(0.5 point) Since f, g preserve orders, we know that they map intervals to intervals and are locally continuous . Now consider the limit $\lim_{x\to r(I)} f(x) = M$. This means that we can take a small interval (a, M) which will then be mapped by f^{-1}, g^{-1} into intervals $(f^{-1}(a), r(I)), (g^{-1}(a), r(j))$ because we can take a close enough to M such that f, g are continuous on $(f^{-1}(a), r(I)), (g^{-1}(a), r(j))$.(0.5 point)

We can now define our function

$$h : (f^{-1}(a), r(I)) \to (g^{-1}(a), r(I) \qquad x \mapsto g^{-1}(f(x)).$$

Note that g^{-1} and f are order preserving, so h must preserve orders as well, therefore h is injective, continuous and maps intervals to intervals (0.5 points). In particular h is surjective since $h(f^{-1}(a)) = g^{-1}(a)$ and $\lim_{x\to r(I)} h(x) = \lim_{x\to M} g^{-1}(x) = r(J)$. This implies that h is a bijection, and since it is order preserving it is both continuous and open, so an homeomorphism (0.5 point).