Tame Topology and O-minimal Structures-Dimensions, Homework Set

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In the following exercises we fix an O-minimal structure $(R, <, \mathcal{S})$:

Exercise 1: (**3 points**) (Dimensions of Sets from Definable Families)
Let A and B be definable subsets of R^{m+n}, with A non-empty. Assume that, for every $a \in R^m$, $\dim(B_a) < \dim(A_a)$. Prove that $\dim B < \dim A$. (For definition of A_a and B_a, check p.59 **(3.1)**.)

Solution: By **(1.6)**Corollary (i), $\dim B = \max_{0 \leq d \leq n}(\dim(B(d)) + d)$. Let d' the value that reaches the maximum, i.e., $\dim B = \dim(B(d')) + d'$. For any $a \in B(d')$, we have $d' = \dim(B_a) < \dim(A_a)$. Thus, $B(d') \subseteq \bigcup_{d'+1 \leq d \leq n} A(d)$. Therefore, $\dim B(d') \leq \max_{d'+1 \leq d \leq n} \dim(A(d))$. And $\dim B = \dim B(d') + d' \leq \max_{d'+1 \leq d \leq n} \dim(A(d)) + d' < \max_{d'+1 \leq d \leq n}(\dim(A(d)) + d) \leq \dim A$.

Exercise 2: (Local Dimension, p.69 **(1.17)** Exercise 2, 3, 4.)

1. (**2 points**) Let $A \subseteq R^m$ be definable and $a \in R^m$. Show there is a number $d \in \{-\infty, 0, \cdots, \dim A\}$ such that there is an open box $U \subseteq R^m$ with $a \in U$, and for all open box $V \subseteq R^m$, if $a \in V$ and $V \subseteq U$, then $\dim(V \cap A) = d$.

   Remark: The number d defined by this property is called the **local dimension of** A **at** a**,** notation $\dim_a(A)$. Note that $\dim_a(A) = -\infty$ iff $a \notin \text{cl}(A)$.
   Solution: For any $i \in \{-\infty, 0, \cdots, \dim A\}$, let $B_i = \{U \subseteq R^m : U \text{ is an open box}, a \in U, \dim(U \cap A) = i\}$. Let d be the minimum i such that $B_i \neq \emptyset$. (Such i exists since for any open box U with $a \in U$, $\dim(U \cap A) \leq \dim A$, hence it cannot be the case that all B_i are empty.) Pick any $U \in B_d$, for all open box $V \subseteq U$ with $a \in V$, $\dim(V \cap A) \leq \dim(U \cap A) = d$ (since $V \cap A \subseteq U \cap A$). Let $d' = \dim(V \cap A)$, then $d' \geq d$, for $B_{d'} \neq \emptyset$. Therefore, $\dim(V \cap A) = \dim(U \cap A) = d$.

2. (**2 points**) Show that if $A \subseteq R^m$ is a d-dimensional cell, then $\dim_a(A) = d$ for all $a \in \text{cl}(A)$.
   Solution: Note that for any $a \in \text{cl}(A)$, for any open box $U \subseteq R^m$ with $a \in U$,

$U \cap A \neq \emptyset$. Let $p_i : R^m \to R^d$ defined as in **(2.7)**, then $p_i : A \to p_i(A)$ is a homeomorphism. Since $U \cap A$ is open and nonempty in A, $p_i(U \cap A)$ must also be open and nonempty in $p_i(A)$. Note that $p_i(A)$ is open in R^d, hence, $p_i(U \cap A)$ is also open in R^d, together with nonemptiness, we conclude that $p_i(U \cap A)$ contains an open box in R^d. Therefore, $\dim(U \cap A) = \dim(p_i(U \cap A)) = d$.

3. (**3 points**) Let $A \subseteq R^m$ be a definable set and $d \in \{0, \cdots, \dim A\}$. Show that the set $\{a \in R^m : \dim_a(A) \geq d\}$ is a definable closed subset of $\mathrm{cl}(A)$. (**Hint:** apply cell decomposition theorem to $\mathrm{cl}(A)$, then show the set $\{a \in R^m : \dim_a(A) \geq d\}$ is the closure of a finite union of cells.)

   Show also that if $A \neq \emptyset$, then $\dim(\{a \in \mathrm{cl}(A) : \dim_a(A) < d\}) < d$.

   Solution: (I am sorry, I think I gave a misleading hint, it would be better to apply cell decomposition to A rather than $\mathrm{cl}(A)$, I will give both the answers.)
   (**Apply to cl**(A)**:**) Let $\mathcal{D}$ be a finite partition of $\mathrm{cl}(A)$ into cells. Let

$$B = \bigcup \{C \in \mathcal{D} : \dim C \geq d\}.$$

   Clearly $\mathrm{cl}(B)$ is a definable closed subset of $\mathrm{cl}(A)$. Claim:

$$\{a \in R^m : \dim_a(A) \geq d\} = \mathrm{cl}(B).$$

   For any $a \in \mathrm{cl}(B)$, $a \in \mathrm{cl}(C)$ for some $C \in \mathcal{D}$ with $\dim C \geq d$. For any open box $U \subseteq R^m$, $a \in U$, $\dim(U \cap \mathrm{cl}(A)) \geq \dim(U \cap C) = \dim C \geq d$. And since for any open box $U \subseteq R^m$, $\mathrm{cl}(A) = \mathrm{cl}(A \cap U) \cup \mathrm{cl}(A \cap (R^m \setminus U))$ and
   $\mathrm{cl}(A \cap (R^m \setminus U)) \subseteq \mathrm{cl}(R^m \setminus U) = R^m \setminus U$, hence

$$U \cap \mathrm{cl}(A) = (\mathrm{cl}(A \cap U) \cap U) \cup (\mathrm{cl}(A \cap (R^m \setminus U)) \cap U) = \mathrm{cl}(U \cap A) \cap U \subseteq \mathrm{cl}(U \cap A).$$

   Therefore, for any box $U \subseteq R^m$, $a \in U$, $d \leq \dim(U \cap \mathrm{cl}(A)) \leq \dim(\mathrm{cl}(U \cap A)) = \dim(U \cap A)$ (last equality by Theorem (1.8)). We conclude that $\dim_a(A) \geq d$.

   On the other hand, for all $a \in \mathrm{cl}(A)$, if $a \notin \mathrm{cl}(B)$, then there is an open box $U \subseteq R^n$, such that $a \in U$ and $U \cap B = \emptyset$. Note that $U \cap A \subseteq \mathrm{cl}(A) \setminus B$ and $\mathrm{cl}(A) \setminus B$ is a finite union of cells with dimension less than d. Hence,
   $\dim(U \cap A) \leq \dim(\mathrm{cl}(A) \setminus B) < d$. And we conclude, $\dim_a(A) < d$.

   From what we have proved before,

$$\{a \in \mathrm{cl}(A) : \dim_a(A) < d\} = \mathrm{cl}(A) \setminus \{a \in R^m : \dim_a(A) \geq d\}$$

   is definable. And

$$\{a \in \mathrm{cl}(A) : \dim_a(A) < d\} = \mathrm{cl}(A) \setminus \mathrm{cl}(B) \subseteq \mathrm{cl}(A) \setminus B.$$

Hence, $\dim(\{a \in \text{cl}(A) : \dim_a(A) < d\}) \le \dim(\text{cl}(A) \setminus B) < d$.
(You can also prove that for any $a \in \text{cl}(A)$, $\dim_a(A) = \dim_a(\text{cl}(A))$, by using $\dim(U \cap \text{cl}(A)) \le \dim(\text{cl}(U \cap A)) = \dim(U \cap A)$ for open box U.)

(**Apply to** A**:**) Let $\mathcal{D}$ be a finite partition of A into cells. Let

$$B = \bigcup \{C \in \mathcal{D} : \dim C \ge d\}.$$

Clearly $\text{cl}(B)$ is a definable closed subset of $\text{cl}(A)$. Claim:

$$\{a \in R^m : \dim_a(A) \ge d\} = \text{cl}(B).$$

For any $a \in \text{cl}(B)$, $a \in \text{cl}(C)$ for some $C \in \mathcal{D}$ with $\dim C \ge d$. For any open box $U \subseteq R^m$, $a \in U$, $\dim(U \cap A) \ge \dim(U \cap C) = \dim C \ge d$. We conclude that $\dim_a(A) \ge d$.

On the other hand, for all $a \in \text{cl}(A)$, if $a \notin \text{cl}(B)$, then there is an open box $U \subseteq R^n$, such that $a \in U$ and $U \cap B = \emptyset$. Note that $U \cap A \subseteq A \setminus B$ and $A \setminus B$ is a finite union of cells with dimension less than d. Hence,
$\dim(U \cap A) \le \dim(A \setminus B) < d$. And we conclude, $\dim_a(A) < d$.

From what we have proved before,

$$\{a \in \text{cl}(A) : \dim_a(A) < d\} = \text{cl}(A) \setminus \{a \in R^m : \dim_a(A) \ge d\}$$

is definable. And

$$\{a \in \text{cl}(A) : \dim_a(A) < d\} = \text{cl}(A) \setminus \text{cl}(B) = (\bigcup_{C \in \mathcal{D}} \text{cl}(C)) \setminus (\bigcup_{C \in \mathcal{D}, \dim(C) \ge d} \text{cl}(C)) = \bigcup_{C \in \mathcal{D}, \dim(C) < d} \text{cl}(C)$$

Hence, $\dim(\{a \in \text{cl}(A) : \dim_a(A) < d\}) \le \dim(\bigcup_{C \in \mathcal{D}, \dim(C) < d} \text{cl}(C)) = \max\{dim(\text{cl}(C)) : C \in \mathcal{D}, \dim(C) < d\} = \max\{dim(C) : C \in \mathcal{D}, \dim(C) < d\} < d$.