seminar o-minimal structures,
solution to hand-in exercise 8
Felix Denis

1

a. Define $S = \{ \ (x_1, \ldots, x_{d-1}, y) \mid y = x_1 \vee \ldots \vee y = x_{d-1} \ \}$. Now let n be arbitrary. Next we pick distinct $y_1, \ldots, y_n \in R$ and let $F = \{y_1, \ldots, y_n\}$. We can easily verify that $\mathcal{C} = \{ \ S_x \mid x \in R^{d-1} \ \}$ cuts out precisely the subsets F with strictly less than d elements. Hence $|\mathcal{C} \cap F| = p_d(n)$ and therefore $f_{\mathcal{C}}(n) \geq p_d(n)$. On the other hand, we have $f_{\mathcal{C}}(d) < 2^d$, because every fiber S_x with $x \in R^{d-1}$ has cardinality $< d$, which implies that for a d-element set $F \subseteq R$ the entire set F itself is not cut out by $\mathcal{C}$. Hence we also have $f_{\mathcal{C}}(n) \leq p_d(n)$ for every n.

b.[1] We assume $(R, <, \mathcal{S})$ expands an ordered abelian group $(R, <, +, 0)$. For notational convenience we pick some $r > 0$ and identify the subgroup of R generated by r with a copy of the ordered group of integers $\mathbb{Z}$. Define a sequence of "triangular" subsets of $R \times R$ as follows:

$$T^{(n)} = \{ \ (x, y) \mid y < n, x < \sum_{i=1}^{n} i, x + y > \sum_{i=1}^{n} \}i$$

Define furthermore for every n the set $S^{(n)} = T^{(1)} \cup \cdots \cup T^{(n)}$, and lastly $\mathcal{D}^{(n)} = \{ \ T_x^{(n)} \mid x \in R \ \}$ and $\mathcal{C}^{(n)} = \{ \ S_x^{(n)} \mid x \in R \ \}$.

First note that for every n, the sets $T^{(n)}$ and $S^{(n)}$ are definable and every nonempty fiber $S_x^{(n)}$ with $x \in R$ coincides with $T_x^{(k)}$ where $0 < k \leq n$ is the unique integer such that $\sum_{i=1}^{k-1} i < x < \sum_{i=1}^{k} i$. So $\mathcal{C}^{(n)} = \mathcal{D}^{(1)} \cup \cdots \cup \mathcal{D}^{(n)}$.

Second, for every $k \in \mathbb{N}$, if we have $0 < y_1 < \ldots < y_m < k$ and we put $F = \{y_1, \ldots, y_m\}$, then the nonempty subsets in $\mathcal{D}^{(k)} \cap F$ are precisely the sets of the form $\{y_i, y_{i+1}, \ldots, y_m\}$ with $1 \leq i < m$.

Now consider $S^{(m)}$, and let $n \geq m$ be arbitrary. Let $y_1, \ldots, y_n$ such that $m > y_1 > m - 1 > y_2 > m - 2 > \ldots > y_{m-1} > 1 > y_m > y_{m+1} \ldots > y_n$ and let $F = \{y_1, \ldots, y_n\}$. Then using the previous remarks it is straightforward to show that the nonempty subsets in $\mathcal{C}^{(m)} \cap F$ are precisely the sets of the form $\{y_i, y_{i+1}, \ldots, y_{n-m+j}\}$ with $1 \leq i \leq n - m + j$ and $1 \leq j \leq m$. The total number of such sets is $n + (n-1) + \cdots + (n - m + 1) = mn - \sum_{i=0}^{m-1} i$. Hence $f_{\mathcal{C}^{(m)}}(n) \geq mn + 1 - \sum_{i=0}^{m-1} i$ for every $n \geq m$.

Finally, given an arbitrary positive number c we can find $m > c$ and $N \geq m$ such that for every $n \geq N$ we get $cn < mn + 1 - \sum_{i=0}^{m-1} i$. Then for $S = S^{(m)}$ and $\mathcal{C} = \mathcal{C}^{(m)}$ we get $f_{\mathcal{C}}(n) > cn$ for every $n \geq N$.

2

a. Since every decomposition of R into distinct cells $E_1, \ldots, E_k$ is in particular a partition of R, every intersection $\bigcap_{1 \leq i \leq k} E_i^{\varepsilon(i)}$ is nonempty precisely when there is exactly one i for which $\varepsilon(i) = 1$, in which case the intersection equals E_i. Hence the atoms are precisely $E_1, \ldots, E_k$.

b. A definably connected subsets of R consists of an interval possibly together with either or both of its endpoints. Given definably connected $S_1, \ldots, S_k \subseteq$

[1] I personally think Martijn's example was the simplest and easier to understand.

R, let $x_1, \dots, x_\ell$ list their endpoints in R in ascending order. Each S_i contributes at most two endpoints to this list, so $\ell \leq 2k$. The decomposition $\{ \ (-\infty, x_1), \{x_1\}, (x_1, x_2), \{x_2\}, \dots, \{x_\ell\}, (x_\ell, +\infty) \ \}$ clearly partitions each of the S_i, and has $2\ell + 1 \leq 4k + 1$ elements.

c. It is sufficient to show that there are c and N such that $f^{\mathcal{G}}(n) \leq cn$ for $n \geq N$. By the finiteness theorem there exists a natural number e such that each cofiber S^y has at most e definably connected components. Let n be arbitrary and suppose $y_1, \dots, y_n \in R^q$ are distinct. Let $I_1, \dots, I_k$ list all the distinct definably connected components of all the S^{y_i}, and note that $k \leq en$. Apply (b) to obtain a decomposition of R into cells $E_1, \dots, E_m$ partitioning each I_i with $m \leq 4k+1$. Since every S^{y_i} is a union of some I_j's, and therefore, of some E_j's, the boolean algebra $B(S^{y_1}, \dots, s^{y_n})$ is contained in $B(E_1, \dots, E_m)$. Hence the number of atoms of the former is bounded by the number of atoms of the latter, which because of (a) is precisely m. Finally, $m \leq 4k + 1 \leq 4en + 1$, and since n and the y_i were arbitrary, we see that $f^{\mathcal{G}}(n) \leq 4en + 1$. We conclude by taking some $c > 4e$ and finding a suitably large N such that $4en + 1 \leq cn$ for $n \geq N$.