Topological Models of Computability

Jaap van Oosten

Department of Mathematics, Utrecht University

Nederlands Mathematisch Congres 53, 11/4/2017
Joint work with Niels Voorneveld

Turing (1936): what is a computation (on natural numbers)?
A human (or a device), after reading a symbol, may perform one of a very limited number of actions:

- he/she may erase the symbol, or replace it with another, and move to the next (or previous) symbol;
- he/she may do nothing, and move to the next (previous) symbol;
- he/she may decide to terminate (halt) the computation.

The computer is guided by the "state of mind" he/she finds himself/herself in.
Turing argued that there can be only finitely many symbols and finitely many states of mind.

A formalisation: the *Turing machine*, a finite automaton.
At the start of the computation there is 'input', written on a tape (the 'input' is a natural number, or a sequence of natural numbers); when the computation is halted, what is on the tape is regarded as 'output' (a natural number).
A partial function $F : \mathbb{N}^k \to \mathbb{N}$ is *partial (Turing) computable* if there is a Turing machine Q such that for all k-tuples $a_1, \ldots, a_k \in \mathbb{N}$ the following holds:
if $F(a_1, \ldots, a_k)$ is defined then computation with Q and input $a_1, \ldots, a_k$ terminates, and yields output $F(a_1, \ldots, a_k)$.

The *Church-Turing Thesis* states that any partial function that is algorithmically computable (by whatever kind of algorithm) is Turing computable.

This thesis has stood the test of time.

For computation on natural numbers, therefore, Turing machines are the definitive answer to the question: what is computability?

However, for computations on higher-type objects (functions of integers, sets of integers), there are various reasonable answers.

Fix a 'reasonable' numbering of all Turing machines: $Q_0, Q_1, \ldots$
Define a partial function $n, m \mapsto nm$ on $\mathbb{N}$:
nm, if defined, is the output of the computation of Turing machine Q_n with input m (if that computation halts).
We can now consider expressions built up from variables, juxtaposition, and brackets: e.g., $(xy)z$, $(xz)(yz)$.
Reasonable convention: abbreviate $(xy)z$ to xyz (so: $(xz)(yz)$ to $xz(yz)$).
Every expression $E(x_1, \ldots, x_n)$ in variables $x_1, \ldots, x_n$ defines a partial function on $\mathbb{N}^k$: $a_1, \ldots, a_k \mapsto E(a_1, \ldots, a_k)$.

Basic Fact: for any expression $E(x_1, \ldots, x_n)$ there is a number m such that for all n-tuples $a_1, \ldots, a_n$ we have:

i) $ma_1 \cdots a_{n-1}$ is defined;

ii) If $E(a_1, \ldots, a_n)$ is defined, then $ma_1 \cdots a_n = E(a_1, \ldots, a_n)$

Note: not every variable x_i has to occur in E.

In particular there is some m such that:

i) ma is defined.

ii) $mab = a$.

Models of Computability

A *Model of Computability* (MoC) is a set X together with a partial binary operation on X, $x, y \mapsto xy$, for which the **Basic Fact** holds.
Some facts: in every MoC X there are elements (*Booleans* and *Definition by Cases*) T, F, C such that

$$\begin{array}{rcl} \mathsf{CT}ab & = & a \\ \mathsf{CF}ab & = & b \end{array}$$

In every MoC there are elements p, p_1, p_2 satisfying

$$p_1(pab) = a \quad p_2(pab) = b$$

Think of pab as a *coded pair*; we denote pab by $[a, b]$.
In every MoC there is a choice of elements $\bar{n}$ (for every natural number n), which system satisfies the following: for every Turing computable function f of k variables, there is some $a_f \in X$ such that whenever $f(a_1, \ldots, a_k)$ is defined, then

$$a_f \bar{a}_1 \cdots \bar{a}_k = \overline{f(a_1, \ldots, a_k)}$$

MoC's are structured in a category (in fact, a 2-category):
For MoC's X, Y, a *morphism* $\gamma : X \to Y$ assigns to every $x \in X$ a nonempty subset $\gamma(x)$ of Y, in such a way that for some element $r \in Y$ we have:
whenever ab is defined in X and $u \in \gamma(a), v \in \gamma(b)$, then ruv is defined in Y, and $ruv \in \gamma(ab)$.
So, *ru simulates* the action of a in Y.
We also require that there is some $d \in Y$ which satisfies:
$u \in \gamma(\mathrm{T}) \Rightarrow du = \mathrm{T}$ and $v \in \gamma(\mathrm{F}) \Rightarrow dv = \mathrm{F}$.

There is always a morphism $\mathbb{N} \to X$: $n \mapsto \bar{n}$.

Example 1

$\mathcal{P}(\mathbb{N})$
Fix a bijection $n, m \mapsto [n, m] : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.
Fix an enumeration $(e_n)_{n\in\mathbb{N}}$ of the finite subsets of $\mathbb{N}$.
Define, for $X, Y \subseteq \mathbb{N}$:

$$XY \; = \; \{n \,|\, \exists m([n, m] \in X \text{ and } e_m \subseteq Y)\}$$

Theorem (Scott) $\mathcal{P}(\mathbb{N})$, equipped with this map, is an MoC.
The *Scott topology* on $\mathcal{P}(\mathbb{N})$ has as basic open sets, sets of the form

$$\mathcal{U}_p \; = \; \{X \subseteq \mathbb{N} \,|\, p \subset X\}$$

for some *finite* $p \subset \mathbb{N}$.
Every map of form $X(-) : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ is Scott-continuous; conversely, every Scott-continuous map is of this form.

Example 2

$\mathbb{N}^{\mathbb{N}}$ (This is the set of all *total* functions from $\mathbb{N}$ to $\mathbb{N}$!)
Fix a bijection $\bigcup_{n\geq 0}\mathbb{N}^n \to \mathbb{N}$: $(a_0,\ldots,a_{n-1}) \mapsto [a_0,\ldots,a_{n-1}]$
Let $\alpha,\beta:\mathbb{N}\to\mathbb{N}$. Say $(\alpha\beta)_n$ *is defined, and equal to* k, if for some number l we have:

$$\begin{array}{rcll} \alpha([n,\beta(0),\ldots,\beta(u-1)]) & = & 0 & \text{for all } u < l \\ \alpha([n,\beta(0),\ldots,\beta(l-1)]) & = & k+1 & \end{array}$$

We now say $\alpha\beta$ is defined, and $\alpha\beta=\gamma$, if for all n, $(\alpha\beta)_n$ is defined, and $\gamma(n)=(\alpha\beta)_n$
Theorem (Kleene) $\mathbb{N}^{\mathbb{N}}$, equipped with this operation, is an MoC.
The *Baire space topology* on $\mathbb{N}^{\mathbb{N}}$ has as basic open sets, sets of the form

$$\mathcal{U}_s \;=\; \{\alpha\,|\,\alpha(0)=s_0,\ldots\alpha(n)=s_n\}$$

for some finite sequence $s=s_0,\ldots,s_n$ of natural numbers.
Every map of form $\alpha(-):\mathbb{N}^{\mathbb{N}}\to\mathbb{N}^{\mathbb{N}}$ is Baire continuous, and conversely.

Oracles

Turing thought about the question: given a (possibly non-computable) function f, what can one compute if an "oracle" gives the value of f at specific arguments?
Computations remain finite, so the oracle can, in one computation, only be consulted a finite number of times.
One gets a preorder on functions: $g \leq_T f$ if g is computable using an oracle for f.

There is a similar preorder on endofunctions on an MoC.
Let X be an MoC and $f : X \to X$ be an arbitrary function.
Define a partial map on $X \times X$, $(a, b) \mapsto a \circ_f b$:
$a \circ_f b = c$ if there is a sequence $u_0, \ldots, u_{i-1}$ of elements of X satisfying:

- either $i = 0$ and $ab = [\mathsf{T}, c]$
- or $i > 0$, and for all $j \leq i - 1$ we have for some $d_j \in X$:

$$abu_0 \cdots u_{j-1} = [\mathsf{F}, d_j] \text{ and } u_j = f(d_j)$$

and $abu_0 \cdots u_{i-1} = [\mathsf{T}, c]$.

Now X, with the operation $\circ_f$, is an MoC.
The identity map $x \mapsto \{x\}$ is a morphism from X to $(X, \circ_f)$, and it has a universal property.
We write $X[f]$ for $(X, \circ_f)$.
In $X[f]$, there is an element a_f such that $a_f b = f(b)$ for all $b \in X$ (the function f is "computable in $X[f]$")
We can now define, for endofunctions f, g on X:
$f \leq_T g$ if f is computable in $X[g]$.

A calculation on $\mathcal{P}(\mathbb{N})$

The complement function $C : X \mapsto \mathbb{N} - X$ is certainly *not* Scott continuous, hence not computable in $\mathcal{P}(\mathbb{N})$.

What happens in $\mathcal{P}(\mathbb{N})[C]$?

The *Cantor topology* on $\mathcal{P}(\mathbb{N})$ has as basic open sets

$$\mathcal{U}_p^q \;=\; \{A \subseteq \mathbb{N} \mid p \subseteq A, A \cap q = \emptyset\}$$

for p, q finite.

Proposition. Every endofunction on $\mathcal{P}(\mathbb{N})$ which is Cantor continuous, is computable in $\mathcal{P}(\mathbb{N})[C]$.

Proposition The function Eq of two variables:

$$X, Y \mapsto \begin{cases} \mathrm{T} & \text{if } X = Y \\ \mathrm{F} & \text{otherwise} \end{cases}$$

is computable in $\mathcal{P}(\mathbb{N})[C]$.

So, $\mathrm{Eq} \leq C$. What about the converse?

Lemma There is no partition $\{\mathcal{U}_0, \mathcal{U}_1, \ldots\}$ of $\mathcal{P}(\mathbb{N})$ into countably many sets, such that the complement function C is Scott continuous on each $\mathcal{U}_i$.

Corollary The complement function is not computable in $\mathcal{P}(\mathbb{N})[f]$, for any function f with countable image; in particular, for a characteristic function such as Eq.

Sources

Jaap van Oosten and Niels Voorneveld, *Extensions of Scott's Graph Model and Kleene's Second Algebra*, arXiv:1610.04050v1

For background:
John Longley and Dag Normann, *Higher Order Computability*, Springer 2015