Synthetic Nonstandard Arithmetic

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Infinitesimals in Antiquity

Antiphon (contemporary of Socrates):

If one inscribed any regular polygon, say a square, in a circle, then inscribed an octagon by constructing isosceles triangles in the four segments, then inscribed isosceles triangles in the remaining eight segments, and so on until the whole area of the circle was by this means exhausted, a polygon would thus be inscribed whose sides, in consequence of their smallness, would coincide with the circumference of the circle

Euler, Cauchy:

Euler writes $2\sinh x = (1+\frac{x}{n})^n - (1-\frac{x}{n})^n$ "for infinitely large n". Then n is treated as a classical natural number (which, for example, is either even or odd). Euler then derives the formula

$$\sin z = z\prod_{k=1}^{\infty}(1-\frac{z^2}{k^2\pi^2})$$

Cauchy proved: the pointwisse limit of a convergent sequence of continuous functions is continuous.

A bit of Logic

A *first-order language* consists of:

- a set F of symbols for functions
- a set R of symbols for relations
- a set C of symbols for fixed elements (*constants*)
- logical symbols $\wedge$,$\vee$,$\rightarrow$,$\neg$,$\forall$,$\exists$
- variables $x, y, z, \ldots$
- the equality symbol $=$

Terms and formulas

Every variable and every constant are *terms*. If f is a function symbol (for a function of n arguments) and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

If R is a relation symbol (for a set of n-tuples) and $t_1, \ldots, t_n$ are terms, then $R(t_1, \ldots, t_n)$ is a formula; if t and s are terms then $t = s$ is a formula.
If ϕ and ψ are formulas then $\phi \wedge \psi$, $\phi \vee \psi$, $\phi \rightarrow \psi$ and $\neg\phi$ are formulas; if ϕ is a formula and x a variable, then $\forall x \phi$ and $\exists x \phi$ are formulas.

Models
Given a set M such that for every function symbol f of n arguments we are given a function $f^M : M^n \to M$, for every relation symbol R of n arguments we are given a set $R^M \subseteq M^n$, and for every constant c we are given an element c^M of M. Then M is called a *model* for this language, and we have a straightforward definition of what it means that a formula ϕ *is true in* M *for elements* $a_1, \ldots, a_n$ *of* M, denoted:

$$M \models \phi[a_1, \ldots a_n]$$

Example

Consider the language with one binary relation symbol R. We can look at two different models M: M_1 is $\mathbb{N}$, with

$$R^{M_1} = \{(n, m) : n \leq m\}$$

and M_2 is $\mathbb{N}$ with

$$R^{M_2} = \{(n, m) : n | m\}$$

Then $M_1 \models R(x, y)[0, 1]$; $M_2 \models \neg R(x, y)[0, 1]$. In both models, the formulas $\forall x R(x, x)$ and $\forall x \forall y \forall z((R(x, y) \wedge R(y, z)) \rightarrow R(x, z))$ are true (both relations are partial orders); moreover,

$$M_1 \models \forall x \forall y (R(x, y) \vee R(y, x))$$

but this does not hold for M_2.

A language for the real numbers

Consider the language $L_{\mathbb{R}}$ which has a function symbol f for every function $f : \mathbb{R}^n \to \mathbb{R}$, a relation symbol R for every subset $R \subseteq \mathbb{R}^n$ and a constant a for every $a \in \mathbb{R}$.

Clearly, $\mathbb{R}$ can be made into a model of $L_{\mathbb{R}}$.

Theorem. There is a model $^*\mathbb{R}$ of $L_{\mathbb{R}}$ such that $\mathbb{R} \subset {}^*\mathbb{R}$ and for every formula ϕ of $L_{\mathbb{R}}$ and every $a_1, \ldots, a_n \in \mathbb{R}$ we have

$$\mathbb{R} \models \phi[a_1, \ldots, a_n] \Leftrightarrow {}^*\mathbb{R} \models \phi[a_1, \ldots, a_n]$$

and moreover, in $^*\mathbb{R}$ there is an element c such that $c > n$ for all natural numbers n.

The number c is called a *nonstandard number*. The elements of $\mathbb{R}$ are called *standard numbers*.

Infinitesimals

Let $c \in {}^*\mathbb{R}$ be nonstandard. Then we have $0 < \frac{1}{c} < \frac{1}{n}$ for every natural number $n > 0$. We call $\frac{1}{c}$ an *infinitesimal*.

For $x, y \in {}^*\mathbb{R}$, write $x \simeq y$ for: $|y - x|$ is either 0 or an infinitesimal.

We have compact definitions of continuity and differentiation:

$f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ is continuous at x if for every infinitesimal y, $f(x + y) \simeq f(x)$

$f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ is differentiable at x with derivative $f'(x) = y$ if and only if for every infinitesimal h:

$$\frac{f(x + h) - f(x)}{h} \simeq f'(x)$$

From $^*\mathbb{R}$ back to $\mathbb{R}$

Call $\xi \in {}^*\mathbb{R}$ *bounded* if there is $x \in \mathbb{R}$ such that $\xi < x$.

Theorem For every bounded $\xi \in {}^*\mathbb{R}$ there is a unique $x = \mathrm{st}(\xi) \in \mathbb{R}$ (the *standard part* of ξ), such that $x \simeq \xi$.

Proof. Suppose $\xi > 0$. Let $x = \sup\{y \in \mathbb{R} : y < \xi\}$.

Now suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$. Let $^*f : {}^*\mathbb{R} \to {}^*\mathbb{R}$ be its extension to $^*\mathbb{R}$. We have:

$$f'(x) \ = \ \mathrm{st}(\frac{^*f(x+h) - {}^*f(x)}{h})$$

for any infinitesimal h.

Nonstandard Arithmetic

The language L_{PA} of *Peano Arithmetic* has function symbols for the function $S(x) = x + 1$, and for the functions addition and multiplication. It has one constant 0. Clearly, $\mathbb{N}$ is a model of L_{PA}; it is called the *standard model*.

The order $x < y$ is *defined* by the formula

$$x < y \;\equiv\; \exists z(x + S(z) = y)$$

Similarly as before, we have:

Theorem There is a model ${}^*\mathbb{N}$ for the language L_{PA} such that $\mathbb{N} \subset {}^*\mathbb{N}$ and for every formula ϕ of L_{PA} and every tuple of natural numbers $a_1, \dots, a_n$ we have:

$$\mathbb{N} \models \phi[a_1, \dots, a_n] \Leftrightarrow {}^*\mathbb{N} \models \phi[a_1, \dots, a_n]$$

and moreover, ${}^*\mathbb{N}$ contains a *nonstandard element* c, such that $c > n$ for all natural numbers n.

Proof. Let $^*\mathbb{N}$ be the extension of the subset $\mathbb{N} \subset \mathbb{R}$ to $^*\mathbb{R}$. That is,

$$^*\mathbb{N} \ = \ \{x \in {}^*\mathbb{R} : {}^*\mathbb{R} \models x \in \mathbb{N}\}$$

Since $\mathbb{R} \models \forall x \exists y (x < y \wedge y \in \mathbb{N})$, the same holds for $^*\mathbb{R}$. So $^*\mathbb{N}$ contains nonstandard elements.

Theorem There is no L_{PA} − *formula* ϕ and a tuple $b_1, \ldots, b_n$ of elements of $^*\mathbb{N}$ such that, for every $a \in {}^*\mathbb{N}$ we have:

$$^*\mathbb{N} \models \phi[b_1, \ldots, b_n, a] \Leftrightarrow a \in \mathbb{N}$$

Proof. In ${}^*\mathbb{N}$, the formula

$$\forall y_1 \forall y_2 \cdots \forall y_n(\phi(\vec{y}, 0) \wedge \forall z(\phi(\vec{y}, z) \rightarrow \phi(\vec{y}, S(z))) \\ \rightarrow \forall z \phi(\vec{y}, z))$$

is true.

Corollary 1 (*Overspill*) Given ϕ and $b_1, \ldots, b_n$ as before. If for every $n \in \mathbb{N}$ there is an $m > n$, $m \in \mathbb{N}$ such that ${}^*\mathbb{N} \models \phi[\vec{b}, m]$, then there is a nonstandard number c such that ${}^*\mathbb{N} \models \phi[\vec{b}, c]$.

For, otherwise the formula $\exists y(x < y \wedge \phi(\vec{b}, y))$ would define $\mathbb{N}$.

Corollary 2 (*Underspill*) Given ϕ and $\vec{b}$ as above, if for every nonstandard c there is a nonstandard $d < c$ such that ${}^*\mathbb{N} \models \phi[\vec{b}, d]$, then there is a standard number n such that ${}^*\mathbb{N} \models \phi[\vec{b}, n]$.

Example (application of Overspill):

Let $a \in {}^*\mathbb{N}$ be nonstandard. Define $a^{\frac{1}{\mathbb{N}}}$ by

$$a^{\frac{1}{\mathbb{N}}} = \{b \in {}^*\mathbb{N} : b^n < a \text{ for all } n \in \mathbb{N}\}$$

Note: the exponential function is L_{PA}-definable over ${}^*\mathbb{N}$
Then $a^{\frac{1}{\mathbb{N}}}$ contains $\mathbb{N}$, hence by Overspill it contains a nonstandard element. It is the largest subset of ${}^*\mathbb{N}$ which is closed under $+$ and $\cdot$ and does not contain a.

What does ${}^*\mathbb{N}$ look like as ordered set?

Answer: $\mathbb{N} + (Q \times \mathbb{Z})$ where Q is a dense linear order without end points.

Applications of nonstandard methods (1)
L. van den Dries and K. Schmidt (Inventiones Math. 76, 1984):

Let I an ideal of $K[X_1, \ldots, X_n]$ generated by polynomials of degree $\leq d$. Then there are bounds $B = B(n, d)$ and $C = C(n, d)$ such that:
1. I is prime if and only if for all $f, g \in K[\vec{X}]$ of degree $\leq B$, $fg \in I$ implies $f \in I$ or $g \in I$;
2. $\sqrt{I}$ is generated by polynomials of degree $\leq B$, and for $f \in K[\vec{X}]$, if $f \in \sqrt{I}$ then $f^C \in I$.
Moreover the bounds B and C work for any field K and any ideal I

Applications of nonstandard methods (2)
L. van den Dries and A. Wilkie (Journ. of Algebra 89, 1984):

Let Γ be a group generated by a finite set X. Define a *growth function* $G : \mathbb{N} \to \mathbb{N}$ by: $G(n)$ is the number of elements of Γ which are represented by a word in $X \cup X^{-1}$ of length $\leq n$.
Γ is said to have *polynomial growth* if for some c, d:

$$G(n) \leq cn^d \text{ for } n \text{ sufficiently large}$$

The authors prove a slightly generalized version of *Gromov's Theorem*: if Γ has polynomial growth, it has a nilpotent subgroup of finite index.

Applications of nonstandard methods (3)
L. van den Dries (Foundations of computational mathematics 3, 2003):

Any algorithm for the greatest common divisor function which is based on a primitive recursive definition, is necessarily extremely inefficient.

Applications in Teaching:

J. Keisler has written the text *Elementary Calculus: an Approach using Infinitesimals* (1970's): an elementary text book on Analysis that was actually used at several universities.

Towards "Synthetic Nonstandard Arithmetic"

Synthetic reasoning: using axioms, not 'hardware' of the model.
Reasons:
1. The models ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$ are hard to analyze concretely.
2. More often than not, reasoning in applications is totally independent of the particular model.

Axiom systems for Nonstandard Arithmetic/Analysis:

J. Keisler: recent papers investigating one such axiomatics in the context of 'reverse mathematics' (e.g. *Nonstandard arithmetic and recursive comprehension*, APAL 161, 2010)

Benci and Di Nasso, *Alpha Theory*, Expositiones Math. 21 (2003). Based on their papers on 'numerosities'; work in an extension of ZFC.

For us, the most inspiring axiomatics is *Internal Set Theory*

Edward Nelson: Internal Set Theory (IST)

Extend the language of Zermelo-Fraenkel Set Theory by one extra (unary) relation symbol st(x) for: "x is standard".
A formula which does not contain st is called *internal*.
We employ quantifiers

$$\begin{aligned} \exists^{\mathrm{st}} x \phi &\equiv \exists x (\mathrm{st}(x) \wedge \phi) \\ \forall^{\mathrm{st}} x \phi &\equiv \forall x (\mathrm{st}(x) \rightarrow \phi) \end{aligned}$$

Axioms of IST: Transfer

For ϕ internal the axiom

$$\forall^{\text{st}} x_1 \cdots \forall^{\text{st}} x_n \left(\forall^{\text{st}} y \phi \leftrightarrow \forall y \phi\right)$$

or equivalently

$$\forall^{\text{st}} x_1 \cdots \forall^{\text{st}} x_n \left(\exists^{\text{st}} y \phi \leftrightarrow \exists x \phi\right)$$

This axiom implies that every set which is uniquely defined by an internal formula is standard.

Axioms of IST: Idealisation

For ϕ internal, the axiom

$$\forall^{\mathsf{st}\,\mathrm{fin}} z \exists y \forall x \in z \phi \leftrightarrow \exists y \forall^{\mathsf{st}} x \phi$$

Taking $y \neq x$ for ϕ gives: there is a nonstandard set y.
Moreover:

If S is a standard finite set, then all its elements are standard
Every infinite set contains a nonstandard element
There is a finite set which contains every standard set.

Axioms of IST: Standardisation

For *arbitrary* ϕ, the axiom

$$\forall^{\mathrm{st}} x \exists^{\mathrm{st}} y \forall^{\mathrm{st}} z (z \in y \leftrightarrow z \in x \wedge \phi)$$

Caveat: the usual set formation rules in set theory apply only for internal formulas. For ϕ not internal, the set $\{y \in x : \phi\}$ is not guaranteed to exist. But there is a set which contains exactly the *standard* $y \in x$ which satisfy ϕ. It is denoted ${}^{S}\{z \in x : \phi\}$.

Example of a non-standard proof.

Theorem (De Bruijn and Erdös, 1951): if G is a graph such that every finite subgraph admits a k-colouring, then so does G.
Proof: assume, by Transfer, that G and k are standard. By Idealization, there is a finite subgraph F of G containing all its standard elements; by hypothesis F has a k-colouring f.
Let $g = {}^S f$. Since f takes only standard values, every standard element of G is in the domain of g. By Transfer, every element of G is in the domain of g. Toverify that g is a k-colouring, it suffices, by Idealization, to examine the standard elements, where it agrees with f. This concludes the proof.

Intuitionistic Arithmetic

Intuitionistic logic does not contain the 'tertium non datur' $\phi \vee \neg\phi$, or the rule that if the assumption $\neg\phi$ leads to a contradiction, then ϕ must be true (we can only deduce that $\neg\neg\phi$ is true).

Intuitionistic Arithmetic has the same language as Peano Arithmetic (0,S,$+$,$\cdot$) and the following axioms:
1.$S(x) \neq 0$
2.$S(x) = S(y) \rightarrow x = y$
3.$x + 0 = x$
4.$x + S(y) = S(x + y)$
5.$x \cdot 0 = 0$
6.$x \cdot S(y) = x \cdot y + x$
7.$\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(S(x))) \rightarrow \forall x \phi(x)$
The last is a family of axioms, *induction axioms*.

Although $\phi \vee \neg\phi$ is not part of intuitionistic logic, one can derive certain instances of it, using the induction axioms:

We can prove $\forall x \forall y(x = y \vee x \neq y)$, $\forall x \forall y(x < y \vee x = y \vee x > y)$, in general $\psi \vee \neg\psi$ if ψ is formed using:
propositional connectives $\wedge, \vee, \neg, \rightarrow$ and
bounded quantifiers $\forall x < t$, $\exists x < t$.

However, there are formulas of the form $\forall x \phi$, with one unbounded quantifier (so, ϕ is quantifier-free), for which 'tertium non datur' fails.

Intuitionistic Nonstandard Arithmetic: a first problem

Suppose we reason intuitionistically, and we consider a model of nonstandard arithmetic ${}^*\mathbb{N}$ with nonstandard elements, for which we have full *transfer* in the sense that for any formula $\phi(\vec{x})$ with $\vec{x} \in \mathbb{N}$, we have:

$$\mathbb{N} \models \phi \;\Leftrightarrow\; {}^*\mathbb{N} \models \phi$$

Then we can prove for any formula of the form $\forall y \Phi$ with Φ quantifier-free:

$$\forall y \Phi \vee \neg \forall y \Phi$$

contradicting our earlier conclusions.
For, let a be a nonstandard element of ${}^*\mathbb{N}$. Intuitionistically we can prove $\forall x(\forall y < x \Phi \vee \exists y < x \neg \Phi)$, so either ${}^*\mathbb{N} \models \forall y < a \Phi$ holds, or ${}^*\mathbb{N} \models \neg \forall y \Phi$. The first implies $\mathbb{N} \models \forall y \Phi$ since every standard number is smaller than every nonstandard number; the second implies $\mathbb{N} \models \neg \forall y \Phi$ by transfer.

In the form $\forall^{\mathrm{st}} \vec{x}(A^{\mathrm{st}} \leftrightarrow A)$, the transfer principle implies the tertium non datur for *all* internal formulas.

Some recent work on intuitionistic nonstandard arithmetic/analysis

Palmgren (A constructive approach to nonstandard analysis, APAL 73, 1995) sets up an axiomatic system for intuitionistic arithmetic in all finite types (a system for $\mathbb{N}$ but also for functions $\mathbb{N} \to \mathbb{N}$, functionals $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ and so on).
He adds an infinite element ∞ of type the integers, has a *limit principle*:

$$\forall x^{\sigma} \exists^{\mathsf{st}} y^{0\to\sigma}(x = y\infty)$$

and a principle for 'equality in the limit':

$$\forall^{\mathsf{st}} x^{0\to\sigma} \forall^{\mathsf{st}} y^{0\to\sigma} \left[x\infty = y\infty \leftrightarrow \exists^{\mathsf{st}} k \forall^{\mathsf{st}} n \geq k(xn = yn)\right]$$

Moerdijk (A model for intuitionistic nonstandard arithmetic, APAL 73, 1995) constructs a sheaf model for nonstandard arithmetic in a constructive universe. The internal theory is classical, but $\neg\forall x(\mathrm{st}(x) \vee \neg\mathrm{st}(x))$ holds. The model satisfies the full transfer principle and is, in a nonstandard way, an elementary extension of $\mathbb{N}$.

Moerdijk and Palmgren (Minimal models of Heyting arithmetic (JSL 62, 1997) build further on this and construct ‘minimal models of arithmetic’, that is: models which in a similar sense as Moerdijk 1995, elementarily embed in every other model.
They also consider nonstandard models Their theory contains an explicit axiom for overspill:

$$\forall^{\mathrm{st}} y A(\vec{x}, y) \rightarrow \exists y[\neg \mathrm{st}(y) \wedge \forall u < y A(\vec{x}, u)]$$

Palmgren (Developments in constructive nonstandard analysis, BSL 4, 1998) develops an impressive body of constructive analysis by nonstandard methods.

Avigad and Helzner (Transfer principles in nonstandard intuitionistic arithmetic, Arch. Math. Logic 41, 2002), analyze carefully what forms of transfer can be allowed without forcing classical logic.

Van den Berg, Briseid, Safarik (A functional interpretation for non-standard arithmetic, unpublished, 2011) have an original realizability interpretation for nonstandard arithmetic in all finite types which validates Nelson's Idealization principle:

$$\forall^{\mathrm{st}} x^{\sigma} \exists y^{\tau} \forall i \leq |x| \phi(x_i, y) \rightarrow \exists y^{\tau} \forall^{\mathrm{st}} x^{\sigma} \phi(x, y)$$

which is stronger than Overspill.