Local Operators in the Effective Topos

Jaap van Oosten (joint work with Sori Lee)

Department of Mathematics, Utrecht University

Proofs and Programs, Luminy, February 16, 2012

If $\mathcal{E}$ is a topos and $\mathcal{C}$ is a full subcategory of $\mathcal{E}$ with the properties:

- $\mathcal{C}$ is closed under finite limits in $\mathcal{E}$
- the embedding $\mathcal{C} \to \mathcal{E}$ has a left adjoint which preserves finite limits

then $\mathcal{C}$ is also a topos, and called a *subtopos* of $\mathcal{E}$.
Subtoposes of $\mathcal{E}$ correspond to *local operators on* $\mathcal{E}$.

Why study local operators in the effective topos?

- "Because it's there" (Mallory); subtoposes form an intrinsic piece of structure of the topos
- local operators form a Heyting algebra into which the semilattice of Turing degrees embeds; hence a playground for doing recursion theory
- local operators define new notions of realizability (also for classical theories)

In a topos with a subobject classifier $1 \xrightarrow{\top} \Omega$ (Ω is to be thought of as the 'set of subsets of a one-element set', and $\top$ names the maximal such subset), a *local operator* is a map $j : \Omega \to \Omega$ which satisfies:

i) $\forall pq.(p \to q) \to (jp \to jq)$ (j is *monotone*)

ii) $j\top = \top$ (j *preserves* $\top$)

iii) $\forall p.jjp \to jp$ (j is *idempotent*)

These properties imply: $\forall pq.j(p \wedge q) \leftrightarrow jp \wedge jq$
A local operator can be regarded as a modal operator on the type theory of the topos.

The *effective topos* $\mathcal{E}ff$ (Hyland 1980) is based on indices of partial recursive functions. For $e, x \in \mathbb{N}$ we write ex for $\varphi_e(x)$, the result of applying the e-th partial recursive function to x. We also write $ex{\downarrow}$ for: *ex is defined*, i.e. $\exists y T(e, x, y)$. We employ a primitive recursive coding of pairs $\langle a, b\rangle$ and sequences $\langle a_0, \ldots, a_{n-1}\rangle$.
Let $A, B \subseteq \mathbb{N}$. We write:

$$\begin{array}{rcl} A \wedge B & = & \{\langle a, b\rangle \,|\, a \in A, b \in B\} \\ A \rightarrow B & = & \{e \,|\, \text{for all } a \in A, ea{\downarrow} \text{ and } ea \in B\} \end{array}$$

This is the *logic of realizability*

The effective topos (continued)

Objects of $\mathcal{E}\mathit{ff}$: pairs $(X, [\![\cdot = \cdot]\!])$ where X is a set and for $x, y \in X$, $[\![x = y]\!]$ is a subset of $\mathbb{N}$ such that the sets

$$\bigcap_{x,y \in X} [\![x = y]\!] \rightarrow [\![y = x]\!]$$
$$\bigcap_{x,y,z \in X} ([\![x = y]\!] \wedge [\![y = z]\!]) \rightarrow [\![x = z]\!]$$

are nonempty.

An arrow $(X, [\![\cdot = \cdot]\!]) \rightarrow (Y, [\![\cdot = \cdot]\!])$ is represented by a function $F : X \times Y \rightarrow \mathcal{P}(\mathbb{N})$ which satisfies conditions...

In $\mathcal{E}\!f\!f$, the subobject classifier is $1 \xrightarrow{\top} \Omega$ where:
$1 = (\{*\}, [\![\cdot = \cdot]\!])$ with $[\![* = *]\!] = \mathbb{N}$
$\Omega = (\mathcal{P}(\mathbb{N}), [\![\cdot = \cdot]\!])$ with $[\![A = B]\!] = (A \to B) \wedge (B \to A)$
A *monotone map*: $\Omega \to \Omega$ is given by a function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ for which

$$E_m(f) \;=\; \bigcap_{p,q \subseteq \mathbb{N}} (p \to q) \to (fp \to fq)$$

is a nonempty set.
Define also:

$$\begin{array}{rcr} E_\top(f) & = & f(\mathbb{N}) \\ E_{\mathrm{id}}(f) & = & \bigcap_{p \subseteq \mathbb{N}} (ffp \to fp) \end{array}$$

A local operator $\Omega \to \Omega$ is given by a function $f : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ for which $E_m(f)$, $E_\top(f)$ and $E_{\mathrm{id}}(f)$ are nonempty. Let

$$E_{\mathrm{loc}}(f) \ = \ E_m(f) \wedge E_\top(f) \wedge E_{\mathrm{id}}(f)$$

For monotone maps $f, g : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ we write

$$[\![\, f \leq g \,]\!] \ = \ \bigcap_{p \subseteq \mathbb{N}} fp \to gp$$

There is a function L, acting on monotone maps f, such that $L(f)$ is a monotone map, and there are indices e_1 and e_2 such that

$$\begin{array}{lcl} e_1 & \in & \bigcap_f E_m(f) \to (E_{\mathrm{loc}}(L(f)) \wedge [\![f \le L(f)]\!]) \\ e_2 & \in & \bigcap_{f,g}(E_m(f) \wedge E_{\mathrm{loc}}(g) \wedge [\![f \le g]\!]) \to [\![L(f) \le g]\!] \end{array}$$

$L(f)$ is the local operator *generated by* f.
Theorem (Pitts) The map L can be defined by

$$L(f)(p) \; = \; \bigcap \{ q \subseteq \mathbb{N} \,|\, (\{0\} \wedge p) \subseteq q \text{ and } (\{1\} \wedge fq) \subseteq q \}$$

Suppose $\{f_n \mid n \in A\}$ is an internal (recursive) family of monotone maps indexed by a nonempty set $A \subseteq \mathbb{N}$. That means: for some $e \in \mathbb{N}$ we have

$$\forall n \in A\,(en \in E_m(f_n))$$

Then the *join* $\bigvee_{n\in A} f_n$ is given by

$$(\bigvee_{n\in A} f_n)(p) \;=\; \{\langle n,x\rangle \mid x \in f_n(p)\}$$

We have for arbitrary monotone g: $\bigcap_{n\in A}(\{n\} \to [\![\, f_n \leq g \,]\!])$ is nonempty if and only if $[\![\, \bigvee_{n\in A} f_n \leq g \,]\!]$ is nonempty.

Let $\mathcal{A}$ be a *nonempty* subset of $\mathcal{P}(\mathbb{N})$ (we write $\mathcal{A} \in \mathcal{P}^*\mathcal{P}(\mathbb{N})$).
Define:

$$G_{\mathcal{A}}(p) \ = \ \bigcup_{A \in \mathcal{A}} (A \to p)$$

Then $G_{\mathcal{A}}$ is monotone. $G_{\mathcal{A}}$ is the least f such that $\bigcap_{A \in \mathcal{A}} f(A) \neq \emptyset$
Every nontrivial monotone map f is a recursive join of such $G_{\mathcal{A}}$:
let $A = \bigcup_{p \subseteq \mathbb{N}} f(p)$ and for $n \in A$ let $f_n = G_{\{q \subseteq \mathbb{N} \,|\, n \in fq\}}$.
Then $f \simeq \bigvee_{n \in A} f_n$

Every local operator $j : \Omega \to \Omega$ satisfies

$$\mathrm{id}_\Omega \leq j \leq \lambda p.\top$$

We call $\lambda p.\top$ the *trivial* local operator (it corresponds to the degenerate topos).

Known results about local operators in $\mathcal{E}ff$:

1. There is the 'double negation' local operator $\neg\neg$:

$$\neg\neg p \;=\; \begin{cases} \mathbb{N} & \text{if } p \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

2. For a monotone map f we have:

a. $L(f)$ is trivial if and only if $f(\emptyset) \neq \emptyset$

b. $L(f)$ is isomorphic to $\neg\neg$ if and only if $f(\emptyset) = \emptyset$ and $L(f)(\{0\}) \cap L(f)(\{1\}) \neq \emptyset$

More known results:
(Pitts) Let $\mathcal{A} = \{\{m \mid m \geq n\} \mid n \in \mathbb{N}\}$. Then $\mathrm{id} < L(G_{\mathcal{A}}) < \neg\neg$
For an arbitrary function $\alpha : \mathbb{N} \to \mathbb{N}$ let $\rho(n) = \{\{\alpha(n)\}\}$
Then for $j = L(\bigvee_{n \in \mathbb{N}} G_{\rho(n)})$ we have that

$$\bigcap_{n \in \mathbb{N}} \{n\} \to j(\{\alpha(n)\}) \neq \emptyset$$

(this means that the function α determines a total map from N to N in the topos corresponding to j), and j is the *least* local operator with this property. Let us denote j by j_α.
Theorem (Hyland) For $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ we have: $j_\alpha \leq j_\beta$ if and only if $\alpha \leq_T \beta$ (α is Turing reducible to β)

Back to monotone maps. Such f can be written as $f = \bigvee_{n \in B} G_{\theta(n)}$ for $\theta : B \to \mathcal{P}^*\mathcal{P}(\mathbb{N})$
We wish to study the map $L(f)$:

$$L(f)(p) \;=\; \bigcap \{q \subseteq \mathbb{N} \,|\, (\{0\} \wedge p) \subseteq q \text{ and } (\{1\} \wedge fq) \subseteq q\}$$

Equivalently, $L(f)(p) = L'(f)(p)_{\omega_1}$ where

$$\begin{array}{rcl} L'(f)(p)_0 & = & \{0\} \wedge p \\ L'(f)(p)_{\alpha+1} & = & L'(f)(p)_\alpha \cup (\{1\} \wedge f(L'(f)(p)_\alpha)) \\ L'(f)(p)_\lambda & = & \bigcup_{\beta<\lambda} L'(f)(p)_\beta \text{ for } \lambda \text{ a limit} \end{array}$$

Definition A *sight* is, inductively,

either a thing called NIL

or a pair (A, σ) with $A \subseteq \mathbb{N}$ and σ a function on A such that $\sigma(a)$ is a sight for each $a \in A$.

To any sight S we associate a well-founded tree $\mathrm{Tr}(S)$ of coded sequences of natural numbers, as well as a subset of its set of leaves (which we call *good leaves*), by induction:

If $S = \text{NIL}$ then $\mathrm{Tr}(S) = \{\langle\rangle\}$ and $\langle\rangle$ is a good leaf;

if $S = (\emptyset, \emptyset)$ then $\mathrm{Tr}(S) = \{\langle\rangle\}$ and $\langle\rangle$ is not a good leaf;

if $S = (A, \sigma)$ then $\mathrm{Tr}(S) = \{\langle a\rangle * t \mid a \in A, t \in \mathrm{Tr}(\sigma(a))\}$, and $\langle a\rangle * t$ is a good leaf of $\mathrm{Tr}(S)$ if and only if t is a good leaf of $\mathrm{Tr}(\sigma(a))$

Consider our typical monotone map $f = \bigvee_{n \in B} G_{\theta(n)}$
For $w \in \mathbb{N}$, $p \subseteq \mathbb{N}$ and a sight S, we say that S is (w, θ, p)-*supporting* if:

- whenever s is a good leaf of $\mathrm{Tr}(S)$, $ws \in \{0\} \wedge p$
- whenever $s \in \mathrm{Tr}(S)$ is not a good leaf, $ws = \langle 1, n \rangle$ with $n \in B$ and $\mathrm{Out}(s) \in \theta(n)$ (where $\mathrm{Out}(s) = \{a \mid s * \langle a \rangle \in \mathrm{Tr}(S)\}$)

Theorem $L(f)$ is isomorphic to the function

$$p \mapsto \{w \mid \text{there is a } (w, \theta, p)\text{-supporting sight}\}$$

If $f = G_{\mathcal{A}}$ we can talk about a $(w, \mathcal{A}, p)$-supporting sight S:

- whenever s is a good leaf of $\mathrm{Tr}(S)$, $ws \in \{0\} \wedge p$
- otherwise, $ws = \langle 1, 0 \rangle$ and $\mathrm{Out}(s) \in \mathcal{A}$

Again, $L(f)$ is isomorphic to

$$p \mapsto \{w \mid \text{ there is a } (w, \mathcal{A}, p)\text{-supporting sight}\}$$

In this talk we concentrate on such $f = G_{\mathcal{A}}$. We are interested in the preorder $(\mathcal{P}^*\mathcal{P}(\mathbb{N}), \leq_L)$ where $\mathcal{A} \leq_L \mathcal{B}$ if and only if $L(G_{\mathcal{A}}) \leq L(G_{\mathcal{B}})$

The following are equivalent:

i) $L(G_{\mathcal{A}}) \leq L(G_{\mathcal{B}})$

ii) $G_{\mathcal{A}} \leq L(G_{\mathcal{B}})$

iii) $\bigcap_{A \in \mathcal{A}} L(G_{\mathcal{B}})(A) \neq \emptyset$

iv) There is a number w such that for all $A \in \mathcal{A}$, there is a $(w, \mathcal{B}, A)$-supporting sight.

Example. Suppose $\mathcal{A}, \mathcal{B} \in \mathcal{P}^*\mathcal{P}(\mathbb{N})$, $\mathcal{B}$ has the n-intersection property (for every n-tuple $B_1, \ldots, B_n \in \mathcal{B}$, $B_1 \cap \cdots \cap B_n \neq \emptyset$) and $\mathcal{A}$ has not. Then $L(G_{\mathcal{A}}) \not\leq L(G_{\mathcal{B}})$.

Lemma 1 If $\mathcal{B}$ has the n-intersection property and $S_1, \ldots, S_n$ are sights on $\mathcal{B}$ (for every i, and every $s \in \mathrm{Tr}(S_i)$ which is not a good leaf, $\mathrm{Out}(s) \in \mathcal{B}$), then there is a $d \in \bigcap_{i=1}^{n} \mathrm{Tr}(S_i)$ which is a good leaf of at least one S_i.

Lemma 2 If S and T are two sights on $\mathcal{B}$ and both are $(w, \mathcal{B}, \mathbb{N})$-supporting, then every good leaf of S is also a good leaf of T.

Example (continued) Suppose $\mathcal{B}$ has n-intersection property and $\mathcal{A}$ contains $A_1, \ldots, A_n$ with $A_1 \cap \cdots \cap A_n = \emptyset$.

Suppose $L(G_{\mathcal{A}}) \leq L(G_{\mathcal{B}})$. Then for some w there is, for each $A \in \mathcal{A}$, a $(w, \mathcal{B}, A)$-supporting sight. In particular for each A_i there is a $(w, \mathcal{B}, A_i)$-supporting sight S_i.

By Lemma 1, there is $d \in \bigcap_{i=1}^{n} \mathrm{Tr}(S_i)$ which is a good leaf of some S_i. By Lemma 2, d is a good leaf of every S_i.

It follows that for each i, $wd \in \{0\} \wedge A_i$; so $wd = \langle 0, x \rangle$ with $x \in \bigcap_{i=1}^{n} A_i$; contradiction.

Finitary examples We look at finite collections $\mathcal{A}$ of finite subsets of $\mathbb{N}$ such that $\bigcap \mathcal{A} = \emptyset$ (otherwise, $L(G_{\mathcal{A}}) \simeq \mathrm{id}$), yet for $A_1, A_2 \in \mathcal{A}$, $A_1 \cap A_2 \neq \emptyset$ (otherwise, $L(G_{\mathcal{A}}) \simeq \neg\neg$).
We consider, for $0 < 2m < \alpha < \omega$, the collection

$$\mathcal{O}^{\alpha}_{m} \;=\; \{\beta \subset \{1, \dots, \alpha\} \,|\, |\alpha - \beta| = m\}$$

the collection of 'co-m-tons' in α
Note: for such $\mathcal{O}^{\alpha}_{m}$ we have $\mathcal{O}^{\alpha}_{m} \not\leq_L \mathcal{F}$, where $\mathcal{F}$ is Pitts' example $\{\{m | m \geq n\} \,|\, n \in \omega\}$. For, $\mathcal{F}$ has the k-intersection property for every k.

A few sample results

In $(\mathcal{P}^*\mathcal{P}(\mathbb{N}), \leq_L)$, $\mathcal{O}_1^\omega = \{p \subseteq \mathbb{N} \mid |\mathbb{N} - p| = 1\}$ is an atom, and $\{\{0\}, \{1\}\}$ is a co-atom.

$\ulcorner \frac{\alpha}{m} \urcorner$ is the least number d such that $\mathcal{O}_m^\alpha$ does not have the d-intersection property. Hence, if $\ulcorner \frac{\alpha}{m+1} \urcorner < \ulcorner \frac{\alpha}{m} \urcorner$, then $\mathcal{O}_m^\alpha <_L \mathcal{O}_{m+1}^\alpha$.

Also, $\mathcal{O}_m^{\alpha+m} <_L \mathcal{O}_m^\alpha$

We have an infinity of pairwise distinct finitary local operators.

Recall: for a function $\phi : \mathbb{N} \to \mathbb{N}$ we say 'j forces ϕ to be total' if

$$\bigcap_n \{n\} \to j(\{\phi(n)\})$$

is nonempty.
For $D \subseteq \mathbb{N}$ we say 'j forces D to be decidable' if j forces χ_D (the characteristic function of D) to be total.
Theorem For $0 < 2m < \alpha < \omega$, $L(G_{\mathcal{O}^\alpha_m})$ does not force any non-recursive D to be decidable.

On the other hand, for Pitts' $\mathcal{F} = \{\{m \mid m > n\} \mid n \in \mathbb{N}\}$, $L(G_{\mathcal{F}})$ forces every *arithmetical* D to be decidable.

Idea: induction on arithmetical complexity. Given $A \subseteq \mathbb{N}$ such that $L(G_{\mathcal{F}})$ forces A to be decidable. We consider

$$\exists A \;=\; \{x \mid \exists n \langle x, n\rangle \in A\}$$

The assumption gives us $F_A \in \bigcap_n(\{n\} \to L(G_{\mathcal{F}})(\{\chi_A(n)\}))$

For given x, consider the sequence $\langle F_A(\langle x, 0\rangle), \ldots, F_A(\langle x, n\rangle)\rangle$

We can construct a recursive function H such that for all x, n:

$$\begin{array}{rl} H(x)n \in L(G_{\mathcal{F}})(\{0\}) & \text{if for some } m \leq n,\ \langle x, m\rangle \in A \\ H(x)n \in L(G_{\mathcal{F}}(\{1\}) & \text{otherwise} \end{array}$$

It follows that for each x, $H(x)n \in L(G_{\mathcal{F}})(\{\chi_{\exists A}(x)\})$ for sufficiently large n. That is,

$$H(x) \in G_{\mathcal{F}}(L(G_{\mathcal{F}})(\{\chi_{\exists A}(x)\}))$$

Using $G_{\mathcal{F}} \leq L(G_{\mathcal{F}})$ and $L(G_{\mathcal{F}})L(G_{\mathcal{F}}) \leq L(G_{\mathcal{F}})$ we get the result.

If j is a local operator in $\mathcal{E}\mathit{ff}$ we can look at the interpretation of first-order arithmetic in the subtopos determined by j. This is given by 'j-realizability'. Define the notion 'n j-realizes ϕ' by induction on ϕ as follows:

n j-realizes an atomic ϕ iff ϕ is true;

n j-realizes $\phi \wedge \psi$ iff $n = \langle a, b \rangle$ such that a j-realizes ϕ and b j-realizes ψ

n j-realizes $\phi \rightarrow \psi$ iff for all m such that m j-realizes ϕ, $nm\!\downarrow$ and

$$nm \in j(\{k \mid k \text{ } j\text{-realizes } \psi\})$$

n j-realizes $\exists x \phi(x)$ iff $n = \langle a, b \rangle$ such that b j-realizes $\phi(a)$

n j-realizes $\forall x \phi(x)$ iff for all m, $nm\!\downarrow$ and

$$nm \in j(\{k \mid k \text{ } j\text{-realizes } \phi(m)\})$$

Using j-realizability we can prove:

Theorem If a local operator j forces every arithmetical subset of $\mathbb{N}$ to be decidable, then the subtopos determined by j satisfies true arithmetic.

We have identified a non-Boolean subtopos of $\mathcal{E}\mathit{ff}$ which nevertheless has true arithmetic: the subtopos determined by $L(G_{\mathcal{F}})$. There are others: e.g. determined by j_α where α is some Δ^1_1-complete function.

Using the language of sights we can express j-realizability more concretely in the case $j = L(G_\theta)$ for $\theta : B \to \mathcal{P}^*\mathcal{P}(\mathbb{N})$ as before.

For example, the implication clause:

> n j-realizes $\phi \to \psi$ iff for every m such that m j-realizes ϕ, $nm{\downarrow}$ and there is an (nm, θ, A)-supporting sight S; where $A = \{k \mid k \text{ } j\text{-realizes } \psi\}$

A variation and application to classical realizability (based on ideas of Wouter Stekelenburg and Thomas Streicher)

In *relative realizability* we consider an inclusion $A^\sharp \subset A$ of partial combinatory algebras, such that:

i) the application on $A^\sharp$ is the restriction of the application of A

ii) $A^\sharp$ contains elements k and s satisfying the PCA axioms for both $A^\sharp$ and A

The relative realizability tripos has, in each fibre over a set X, the set of all functions from X to $\mathcal{P}(A)$. The preorder is defined by: $\phi \leq \psi$ iff the set $\bigcap_{x\in X}(\phi(x) \to \psi(x))$ contains an element of $A^\sharp$. Let U be a proper subset of $A - A^\sharp$. Then the map $((-) \to U) \to U : \mathcal{P}(A) \to \mathcal{P}(A)$ defines a nontrivial local operator on the relative realizability topos; the corresponding subtopos is Boolean.

Given a relative realizability situation $A^\sharp \subset A$ and $U \subset (A - A^\sharp)$ we make the following definitions: let $\Lambda = A$ and Π be the set of coded sequences of elements of A. For $s \in \Lambda$ and $\pi = \langle \pi_0, \ldots, \pi_{n-1} \rangle$ we write $s \circ \pi$ for $\langle s, \pi_0, \ldots, \pi_{n-1} \rangle$. We write $\pi_{\geq k}$ for $\langle \pi_k, \ldots, \pi_{n-1} \rangle$. Elements of $\Lambda \times \Pi$ are denoted $s * \pi$. We define a new (total) application on A by: $t \bullet s = \lambda\rho.t(s \circ \rho)$. Define:

$$
\begin{array}{rcl}
\perp\!\!\!\perp & = & \{t * \pi \mid t\pi \text{ is defined and } \in U\} \\
K & = & \lambda\pi.\pi_0(\pi_{\geq 2}) \\
S & = & \lambda\rho.\rho_0(\rho_2 \circ \rho_1(\rho_{\geq 2})) \\
k_\pi & = & \lambda\rho.\rho_0\pi \\
\text{œ} & = & \lambda\rho.\rho_0(k_{\rho_{\geq 1}} \circ \rho_{\geq 1})
\end{array}
$$

We can prove:

$$
\begin{array}{llcl}
(S1) & t * s \circ \pi \in \perp\!\!\!\perp & \Leftrightarrow & t \bullet s * \pi \in \perp\!\!\!\perp \\
(S2) & t * \pi \in \perp\!\!\!\perp & \Leftrightarrow & K * t \circ s \circ \pi \in \perp\!\!\!\perp \\
(S3) & (t \bullet u) \bullet (s \bullet u) * \pi \in \perp\!\!\!\perp & \Leftrightarrow & S * t \circ s \circ u \circ \pi \in \perp\!\!\!\perp \\
(S4) & t * k_\pi \circ \pi \in \perp\!\!\!\perp & \Leftrightarrow & \text{æ} * t \circ \pi \in \perp\!\!\!\perp \\
(S5) & t * \pi \in \perp\!\!\!\perp & \Leftrightarrow & k_\pi * t \circ \pi' \in \perp\!\!\!\perp
\end{array}
$$

This means, that the tuple $(\Lambda, \Pi, \bullet, \perp\!\!\!\perp, \text{æ}, k_{(-)}, K, S)$ is a *strong abstract Krivine structure* in the sense of Streicher. We can let QP (the set of quasi-proofs) be $A^\sharp$.

Paper ‘Basic Subtoposes of the Effective Topos’ (Sori Lee and Jaap van Oosten):
ArXiv:1201.2571v1