Classical and Relative Realizability

Jaap van Oosten

Department of Mathematics, Utrecht University

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In Honour of Dana Scott
Joint work with Tingxiang Zou

Synopsis:

- Some history
- Analysis of Relative realizability
- Streicher-style classical realizability
- Booleanisation of closed subtoposes of relative realizability toposes
- When are such Boolean toposes localic?

History:

- Kleene 1957: gives first definition of relative realizability
- Kleene-Vesley 1965: relative realizability interpretation of intuitionistic analysis
- Late 1990s: revival by Awodey, Birkedal and Scott
- Late 1990s: Krivine discovers Classical realizability
- 1999: JvO visits Pittsburgh, listens to Anne-Sophie Mutter and learns lurid details about Clinton/Lewinsky
- 2002: paper Birkedal-JvO on Relative and Modified Relative Realizability
- 2006: paper Hofstra: All realizability is relative
- 2012: paper Streicher: triposes for classical realizability

BCOs and filtered order-pcas

We start our analysis from the point of view of Hofstra's *Basic Combinatorial Objects*.

A basic combinatorial object (BCO) is a poset $(\Sigma, \leq)$ together with a set $\mathcal{F}_\Sigma$ of partial endofunctions on Σ, with the following properties:

- Each $f \in \mathcal{F}_\Sigma$ has downwards closed domain and is order-preserving on its domain
- There is $i \in \mathcal{F}_\Sigma$ such that for all $x \in \Sigma$, $i(x) \leq x$
- For every pair $f, g \in \mathcal{F}_\Sigma$ there is $h \in \mathcal{F}_\Sigma$ such that whenever $g(f(x))$ is defined, so is $h(x)$, and $h(x) \leq g(f(x))$

A morphism of BCOs $(\Sigma, \leq, \mathcal{F}_\Sigma) \to (\Theta, \leq, \mathcal{F}_\Theta)$ is a function $\phi : \Sigma \to \Theta$ with the properties:

- There is $u \in \mathcal{F}_\Theta$ such that $a \leq a' \in \Sigma$ implies $u(\phi(a)) \leq \phi(a')$ in Θ
- For every $f \in \mathcal{F}_\Sigma$ there is $g \in \mathcal{F}_\Theta$ such that whenever $f(x)$ is defined, $g(\phi(x)) \leq \phi(f(x))$

For two morphisms $\phi, \psi : \Sigma \to \Theta$ we say $\phi \leq \psi$ if for some $g \in \mathcal{F}_\Theta$ we have $g(\phi(x)) \leq \psi(x)$ for all $x \in \Sigma$.

BCOs, morphisms and inequalities form a preorder-enriched category $\mathbb{BCO}$.

$\mathbb{BCO}$ has finite products, so we can talk about BCOs with *internal finite meets* (the BCO maps $\Sigma \to 1$ and $\Sigma \to \Sigma \times \Sigma$ have right adjoints in $\mathbb{BCO}$).

There is a 2-monad $\mathcal{D}$ on $\mathbb{BCO}$: $\mathcal{D}\Sigma$ is the poset of downsets of Σ, with set of partial endomaps defined as follows: $F : \mathcal{D}\Sigma \rightharpoonup \mathcal{D}\Sigma$ is in $\mathcal{F}_{\mathcal{D}\Sigma}$ if and only if for some $f \in \mathcal{F}_\Sigma$ we have:

$U \in \mathrm{dom}(F)$ iff $U \subseteq \mathrm{dom}(f)$

$F(U)$ is the downwards closure of $\{f(x) \,|\, x \in U\}$

The monad $\mathcal{D}$ is a KZ-monad: algebra structures are unique up to isomorphism, and are left adjoint to units.

Every BCO Σ determines a Set-indexed preorder $[-, \Sigma]$: for two functions $\alpha, \beta : X \to \Sigma$, $\alpha \leq \beta$ iff for some $f \in \mathcal{F}_\Sigma$ we have $f(\alpha(x)) \leq \beta(x)$ for all $x \in X$

Example: filtered order-pcas.

An order-pca A is a poset $(A, \leq)$ with a partial (application) map $A \times A \rightharpoonup A$, written $a, b \mapsto ab$, satisfying:

> The domain of the application map is downwards closed and application is order-preserving in both variables on its domain
>
> There are elements k, s in A with $\mathsf{k}xy \leq x$ and $\mathsf{s}xyz \leq (xz)(yz)$ (whenever $xz(yz)$ is defined).

A *filter* on an order-pca is a subset A', closed under the application map, and containing choices for k and s.

Every order-pca A with filter A' is a BCO, with set of endomaps the maps $x \mapsto ax$, for $a \in A'$.

Note that every meet-semilattice $(A, \wedge, \top)$ is a filtered order-pca, with $\wedge$ as application and $\{\top\}$ as filter.

We have a straightforward generalization of Longley's *applicative morphisms* to filtered order-pcas.

Proposition

An applicative morphism of filtered order-pcas is just a BCO map which preserves internal finite meets.

Theorem (Hofstra)

For a BCO Σ, the Set-indexed preorder $[-, \mathcal{D}\Sigma]$ is a tripos precisely when Σ is equivalent to a filtered order-pca.

We are also interested in the question: when is $[-, \Sigma]$ a tripos?

Theorem (vO-Zou)

For a BCO Σ, the Set-indexed preorder $[-, \Sigma]$ is a tripos precisely when Σ is a filtered order-pca and a pseudo $\mathcal{D}$-algebra such that the algebra structure $\bigvee : \mathcal{D}\Sigma \to \Sigma$ preserves internal finite meets.

We could call such filtered order-pcas *lex cocomplete* (after Garner-Lack).
Note: this generalizes the infinite distributivity condition for locales.

Corollary

If $[-, \Sigma]$ is a tripos, it is a subtripos of $[-, \mathcal{D}\Sigma]$.

Classical Realizability

Definition (Streicher)

An abstract Krivine structure *(aks) consists of the following data:*

i) *A set* Λ *of* terms, *together with a binary operation* $t, s \mapsto t \cdot s : \Lambda \times \Lambda \to \Lambda$, *and distinguished elements* $\mathsf{K}, \mathsf{S}, \text{æ}$.

ii) *A subset* QP *of* Λ *(the set of* quasi-proofs*), which contains* K, S *and* æ, *and is closed under the binary operation of i).*

iii) *A set* Π *of* stacks *together with a 'push' operation*

$$t, \pi \mapsto t.\pi : \Lambda \times \Pi \to \Pi$$

as well as an operation

$$\pi \mapsto k_\pi : \Pi \to \Lambda$$

iv) A subset $\perp\!\!\!\perp$ (the *pole*) of $\Lambda \times \Pi$, which satisfies the following requirements:

(S1) If $(t, s.\pi) \in \perp\!\!\!\perp$ then $(t \cdot s, \pi) \in \perp\!\!\!\perp$
(S2) If $(t, \pi) \in \perp\!\!\!\perp$ then $(\mathsf{K}, t.s.\pi) \in \perp\!\!\!\perp$ (for any term s)
(S3) If $((t \cdot u) \cdot (s \cdot u), \pi) \in \perp\!\!\!\perp$ then $(\mathsf{S}, t.s.u.\pi) \in \perp\!\!\!\perp$
(S4) If $(t, k_\pi.\pi) \in \perp\!\!\!\perp$ then $(\text{œ}, t.\pi) \in \perp\!\!\!\perp$
(S5) If $(t, \pi) \in \perp\!\!\!\perp$ then $(k_\pi, t.\pi') \in \perp\!\!\!\perp$ (for any π')

Given a set U of terms and a set α of stacks, we define

$$\begin{array}{rcl} U^{\perp\!\!\!\perp} & = & \{\pi \in \Pi \,|\, \text{for all } t \in U,\ (t,\pi) \in \perp\!\!\!\perp\} \\ \alpha^{\perp\!\!\!\perp} & = & \{t \in \Lambda \,|\, \text{for all } \pi \in \alpha,\ (t,\pi) \in \perp\!\!\!\perp\} \end{array}$$

Let $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ be $\{\beta \subseteq \Pi \,|\, \beta^{\perp\!\!\!\perp\perp\!\!\!\perp} = \beta\}$, ordered by *reverse* inclusion. We define an application $\bullet$ on $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ by putting

$$\alpha \bullet \beta \ = \ \{\pi \in \Pi \,|\, \text{for all } t \in |\alpha| \text{ and } s \in |\beta|,\ (t, s.\pi) \in \perp\!\!\!\perp\}^{\perp\!\!\!\perp\perp\!\!\!\perp}$$

Moreover, let $\Phi \subseteq \mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ be the set

$$\Phi \ = \ \{\alpha \in \mathcal{P}_{\perp\!\!\!\perp}(\Pi) \,|\, \alpha^{\perp\!\!\!\perp} \cap \mathrm{QP} \neq \emptyset\}$$

Theorem (Streicher)

The set $\mathcal{P}_{\perp\!\!\!\perp}(\Pi)$ *forms, together with the given application, a total order-ca, and* Φ *is a filter in it. The Set-indexed preorder* $[-, \mathcal{P}_{\perp\!\!\!\perp}(\Pi)]$ *is a Boolean tripos.*

A tripos of this form is called a *Krivine tripos*.

Given a filtered order-pca (A, A') and a downwards closed set $U \subseteq A - A'$, we can produce an abstract Krivine structure, giving (by Streicher's construction) a filtered order-ca $\mathcal{P}(\Pi)^U_{A,A'}$.

Proposition

The tripos $[-, \mathcal{P}(\Pi)^U_{A,A'}]$ *is equivalent to the subtripos of* $[-, \mathcal{D}(A, A')]$ *given by the local operator* $((-) \Rightarrow U) \Rightarrow U$*.*

Corollary

Every Krivine tripos is the Booleanisation of a closed subtripos of a relative realizability tripos.

Proposition

The tripos $[-, \mathcal{P}(\Pi)^U_{A,A'}]$ *is localic if and only if there exists an element* $e \in A'$ *with the following property:*

$$\text{whenever } ba \in U \text{ for some } b \in A',\ \text{then } ea \in U$$

Example

1. If $U = A - A'$ then $[-, \mathcal{P}(\Pi)^U_{A,A'}]$ is localic.
2. There is a "Turing reducibility" preorder $\leq_T$ on A: $a \leq_T a'$ iff for some $b \in A'$, $ba' \leq a$. Note: $a \leq a'$ implies $a' \leq_T a$. Suppose U is upwards closed w.r.t. $\leq_T$ (hence downwards closed w.r.t. $\leq$). Then $[-, \mathcal{P}(\Pi)^U_{A,A'}]$ is localic.
3. Consider the pca $\mathcal{K}_2 = \mathbb{N}^{\mathbb{N}}$ with filter the set Rec of total recursive functions; let U be a set of non-recursive functions. If U is discrete in the subspace topology of $\mathbb{N}^{\mathbb{N}}$, then $[-, \mathcal{P}(\Pi)^U_{\mathcal{K}_2,\mathrm{Rec}}]$ is non-localic.

Next Project

Can Krivine's models for ZF be constructed as initial ZF-algebras in these boolean subtoposes of relative realizability toposes?