Type Theory and Homotopy Theory

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1. Martin-Löf Type Theory

The world is organized in *types* rather than sets.

Types are *dependent*: that is, given a type A and elements x of A, we may have types $B(x)$, varying with x. And so on.

In the formal system, there are four kinds of basic utterances (*judgements*):

A type (A is a type)

$a : A$ (a is a term - denotes an element - of type A)

$A = B$ (types A and B are equal)

$a = b : A$ (terms a and b are equal as elements of type A)

Every judgement is warranted by a suitable *context*, which is a variable declaration:

$$x_1{:}A_1, \ldots, x_n{:}A_n$$

And there is a system of basic inferences, leading to statements of the form

$$\begin{array}{rcl} x_1{:}A_1, \ldots, x_n{:}A_n & \vdash & A \text{ type} \\ x_1{:}A_1, \ldots, x_n{:}A_n & \vdash & a{:}A \\ x_1{:}A_1, \ldots, x_n{:}A_n & \vdash & A = B \\ x_1{:}A_1, \ldots, x_n{:}A_n & \vdash & a = b{:}A \end{array}$$

Each of these statements expresses that a certain judgement is *valid under context* $x_1{:}A_1, \ldots, x_n{:}A_n$

The formation of contexts is intertwined with the system of inferences. The context

$$x_1{:}A_1, \ldots, x_n{:}A_n$$

is only legitimate provided

$$x_1{:}A_1, \ldots, x_{n-1}{:}A_{n-1} \vdash A_n \text{ type}$$

is a valid judgement.

There is a legitimate *empty context* to start with, and some basic types, for example the natural numbers:

$$\begin{array}{rcl} & \vdash & N \text{ type} \\ & \vdash & 0{:}N \\ x{:}N & \vdash & S(x){:}N \end{array}$$

Given a valid judgement $x_1{:}A_1, \ldots, x_n{:}A_n \vdash B(x_n)$ type, we think of an indexed family of types

$$\{B(x) \,|\, x \in A_n\}$$

(relative to the context $x_1{:}A_1, \ldots, x_{n-1}{:}A_{n-1}$)
We can substitute terms for variables in dependent types: given valid judgements $x_1{:}A_1, \ldots, x_n{:}A_n \vdash B(x_n)$ type and $x_1{:}A_1, \ldots, x_{n-1}{:}A_{n-1} \vdash t(x_1, \ldots, x_{n-1}){:}A_n$ we have

$$x_1{:}A_1, \ldots, x_{n-1}{:}A_{n-1} \vdash B(t) \text{ type}$$

At the moment, we do not know how to construct a single dependent type! Bear with me.
Given a valid judgement

$$x_1{:}A_1, \dots, x_n{:}A_n \vdash B(x_n) \text{ type}$$

one can form the *dependent product* of the indexed family of types: one can infer the judgement

$$x_1{:}A_1, \dots, x_{n-1}{:}A_{n-1} \vdash \prod x{:}A_n . B(x) \text{ type}$$

and there are inference rules governing the behaviour of the product.
From now, we shall abbreviate a context by the symbol Γ; a horizontal line denotes a permitted inference from the hypotheses above to the conclusion below.

Rules for $\prod$:

$$\frac{\Gamma, x{:}A \vdash \ B(x) \text{ type}}{\Gamma \vdash \prod x{:}A.B(x) \text{ type}}$$

$$\lambda\ \frac{\Gamma, x{:}A \vdash t(x){:}B(x)}{\Gamma \vdash \lambda x{:}A.t(x){:} \prod x{:}A.B(x)}$$

$$\text{App}\ \frac{\Gamma \vdash t{:} \prod x{:}A.B(x) \qquad \Gamma \vdash s{:}A}{\Gamma \vdash ts{:}B(s)}$$

$$\beta\ \frac{\Gamma, x{:}A \vdash t(x){:}B(x) \qquad \Gamma \vdash s{:}A}{\Gamma \vdash (\lambda x{:}A.t(x))s = t[s/x]{:}B(s)}$$

(Here $t[s/x]$ denotes substitution of the term s for x in $t(x)$)

$$\eta\ \frac{\Gamma \vdash t{:} \prod x{:}A.B(x)}{\Gamma \vdash (\lambda x{:}A.tx) = t{:} \prod x{:}A.B(x)}$$

Similarly, there is a type construction $\sum x{:}A.B(x)$ for *disjoint sum*. The equality symbol $=$ is sometimes pronounced as *definitional equality*. It allows substitutions:

$$\frac{\Gamma, x{:}A \vdash B(x)\ \text{type} \qquad \Gamma \vdash t = s{:}A}{\Gamma \vdash B(t) = B(s)}$$

and

$$\frac{\Gamma, x{:}A \vdash B(x)\ \text{type} \qquad \Gamma \vdash t = s{:}A \qquad \Gamma \vdash u{:}B(t)}{\Gamma \vdash u{:}B(s)}$$

But, there is a more intriguing notion of "equality", the central notion of this talk: the *Identity type*.

Identity types.

Professor Whitehead writes in his last book that if we begin to ask ourselves the meaning of the simple word "equal" we find ourselves plunged into abstruse modern speculations concerning the character of the universe.
(E. Cunningham)

For any type A and elements x, x' of A we have the type $\mathrm{Id}_A(x, x')$ of proofs that x and x' are identical as elements of A:

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma, x{:}A, x'{:}A \vdash \mathsf{Id}_A(x, x') \text{ type}}$$

We have an axiom

$$\Gamma, x{:}A \vdash \mathsf{refl}_A(x){:}\mathsf{Id}_A(x, x)$$

and two inference rules:

$$\frac{\begin{array}{r}\Gamma, x{:}A, x'{:}A, \alpha{:}\mathsf{Id}_A(x,x') \vdash B(x,x',\alpha) \text{ type} \\ \Gamma, x{:}A \vdash t(x){:}B(x,x,\mathsf{refl}_A(x))\end{array}}{\Gamma x{:}A, x'{:}A, \alpha{:}\mathsf{Id}_A(x,x') \vdash J(x,x',\alpha,t){:}B(x,x',\alpha)}$$

$$\frac{\begin{array}{r}\Gamma, x{:}A, x'{:}A, \alpha{:}\mathsf{Id}_A(x,x') \vdash B(x,x',\alpha) \text{ type} \\ \Gamma, x{:}A \vdash t(x){:}B(x,x,\mathsf{refl}_A(x))\end{array}}{\Gamma, x{:}A \vdash J(x,x,\mathsf{refl}_A(x),t) = t{:}B(x,x,\mathsf{refl}_A(x))}$$

2. Semantics of Type Theory

First, a simple-minded approach, ignoring the identity types.
If, under context Γ, we have $x{:}A \vdash B(x)$ type, we think of a function

$$B \xrightarrow{f} A$$

(interpreting $B(x)$ as $f^{-1}(x)$)
And if under Γ we have $x{:}A, u{:}B(x) \vdash t(u){:}C(x)$ we think of a commutative triangle

$$\begin{tikzcd} B \arrow[rr, "t"] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}$$

We describe our intuition for $\prod$ and $\sum$-types in the following abstract categorical setting:

Locally cartesian closed categories

Given a category $\mathcal{C}$ and an object A, the *slice category* $\mathcal{C}/A$ has as objects: arrows $B \to A$ in $\mathcal{C}$, and as arrows from $B \to A$ to $C \to A$ commutative triangles

$$\begin{tikzcd} B \arrow[rr, "t"] \arrow[dr] & & C \arrow[dl] \\ & A & \end{tikzcd}$$

If $\mathcal{C}$ has pullbacks, then for any arrow $\phi : A \to A'$ we have a pullback functor $\phi^* : \mathcal{C}/A' \to \mathcal{C}/A$

$\mathcal{C}$ is *locally cartesian closed* if $\mathcal{C}$ has pullbacks and every ϕ^* has a *right adjoint*

$$\Pi_\phi : \mathcal{C}/A \to \mathcal{C}/A'$$

Equivalently: every slice $\mathcal{C}/A$ is cartesian closed.

In a locally cartesian closed category we can interpret a type judgement

$$x_1{:}A_1, \dots, x_n{:}A_n \vdash A \text{ type}$$

as a sequence of arrows

$$A \xrightarrow{f} A_n \xrightarrow{f_{n-1}} A_{n-1} \to \cdots \xrightarrow{f_1} A_1$$

Then the judgement

$$x_1{:}A_1, \dots, x_{n_1}{:}A_{n-1} \vdash \prod x{:}A_n.A \text{ type}$$

is interpreted as

$$\Pi_{f_{n-1}}(f) \to A_{n-1} \to \cdots \to A_1$$

But, what to do with the identity types?

3. Homotopy-theoretic semantics Crucial intuition (apparently due to Moerdijk):
Identity types behave like *path spaces*, so there should be an interpretation of type theory in terms of homotopy theory.
Recall: two continuous functions $f, g : X \to Y$ are homotopic if there is a continuous map $H : X \times [0,1] \to Y$ satisfying $H(x,0) = f(x)$ and $H(x,1) = g(x)$.
A continuous function $f : X \to Y$ is a *homotopy equivalence* if there is a continuous function $g : Y \to X$ such that both compositions fg and gf are homotopic to the identity functions.
A *path* in a space X is a continuous function $f : [0,1] \to X$. It is said to be *from* $f(0)$ *to* $f(1)$. Given two paths f and g from x to y a *path homotopy* from f to g is a homotopy H from f to g which keeps the end-points fixed: $H(0,t) = x$ and $H(1,t) = y$.

The space X, together with path homotopy classes of paths, forms a *groupoid*: a category in which every arrow is an isomorphism. It is called the *fundamental groupoid on X*.
Rather than quotienting by the homotopy relation, we can consider X with the following structure:
points ("0-homotopies")
paths ("1-homotopies")
2-homotopies: path homotopies between compositions of paths
and so on.

The n-homotopies can be composed (for $n > 0$). This composition is not associative, but 'associative up to $n+1$-homotopy'.
Similarly, they have inverses 'up to $n+1$-homotopy' and there identities up to $n+1$-homotopy.
This leads to the concept of *weak ∞-groupoid*; we speak of the *fundamental weak ∞-groupoid of the space X*.

In fact, Van den Berg and Garner have shown that “Identity types are weak ∞-groupoids”.

In Type Theory one can prove the following: if B is a type depending on type A (so $x{:}A \vdash B(x)$ type) and we have x, x' in A, u in $B(x)$ and α in $\mathsf{Id}_A(x, x')$, then there is u' in $B(x')$ and β in

$$\mathsf{Id}_{\sum x:A.B(x)}(\langle x, u\rangle, \langle x', u'\rangle)$$

which β projects down to α.

This looks like a 'path-lifting' property.

In topology, a continuous map $E \xrightarrow{p} B$ is called a *(Hurewicz) fibration* if it has precisely such a path-lifting property: given a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "h"'] & B \end{tikzcd}$$

(where $i_0(y) = (y, 0)$)
there is a *diagonal filler*, i.e. a map $\tilde{h} : Y \times [0,1] \to E$ such that $p\tilde{h} = h$ and $\tilde{h}i_0 = f$.

Our discussion suggests: a dependent type $x{:}A \vdash B(x)$ type should be interpreted by a map $B \to A$ which is like a fibration.

Summarizing so far:
we need a locally cartesian closed category with a class of arrows $\mathcal{F}$ (thought of as fibrations) which must satisfy at least:
$\mathcal{F}$ is closed under composition
$\mathcal{F}$ is closed under Π-functors along arrows in $\mathcal{F}$: given

$$\begin{tikzcd} A \arrow[d, "b"'] & \\ B \arrow[r, "f"'] & C \end{tikzcd}$$

with $d, f \in \mathcal{F}$, then $\Pi_f(b) \in \mathcal{F}$
Moreover, $\mathcal{F}$ must have certain 'right lifting properties'.

The category of simplicial sets

Let $\Delta\{[0],[1],[2],\dots\}$ where $[n]=\{0,1,\dots,n\}$ considered as a linearly ordered set.

Δ is a category with as arrows $[n]\to[m]$ the order-preserving functions: $i\le j \Rightarrow f(i)\le f(j)$.

Δ is generated by the maps

$$\begin{array}{lll} d^i : [n-1]\to[n] & (n\ge 1, 0\le i\le n) & \text{omit } i \\ s^i : [n]\to[n-1] & (n\ge 1, 0\le i\le n-1) & \text{double } i \end{array}$$

(these maps satisfy some equations)

A *simplicial set* is a contravariant functor from Δ to the category of sets. A *morphism of simplicial sets* is a natural transformation.

So, a simplicial set X looks like an array of sets:

$$X = (X_0, X_1, \ldots)$$

with functions

$$\begin{array}{ll} d_i : X_n \to X_{n-1} & \textit{face maps} \\ s_i : X_{n-1} \to X_n & \textit{degeneracy maps} \end{array}$$

An element of X_n is called an *n-simplex* of X; it is *nondegenerate* if it is not in the image of some s_i.

The *standard n-simplex* $\Delta[n]$ is the simplicial set generated by one nondegenerate n-simplex. The *geometric n-simplex* is the following subspace of $\mathbb{R}^{n+1}$:

$$\{(x_1, \ldots, x_{n+1}) \,|\, 0 \leq x_1, \ldots, x_{n+1} \leq 1, \sum_{i=1}^{n+1} x_i = 1\}$$

There is a *geometric realization* functor $|\cdot|$ from simplicial sets to topological spaces, obtained by taking, for each nondegenerate n-simplex of a simplicial set X, a copy of the geometric n-simplex, and glue these together according to the equalities between faces which hold in X.

Given the standard n-simplex, its i-th *horn* Λ^i_n is the simplicial set generated by the faces of $\Delta[n]$ except for the one opposite the i-th vertex.

A map $p : E \to B$ of simplicial sets is called a *Kan fibration* if for every commutative diagram

$$\begin{tikzcd} \Lambda^i_n \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ \Delta[n] \arrow[r] & B \end{tikzcd}$$

there is a diagonal filler: $\Delta[n] \to E$.

A map $f : X \to Y$ of simplicial sets is called a *weak equivalence* if its geometric realization $|f| : |X| \to |Y|$ is a homotopy equivalence between topological spaces.

Let us call a map $f : X \to Y$ of simplicial sets *injective* if every component $f_n : X_n \to Y_n$ is an injective function.

We have the following facts:

1. Every map between simplicial sets can be written as the composition of an injective weak equivalence, followed by a Kan fibration;

2. For every commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E \arrow[d, "p"] \\ Y \arrow[r] & B \end{tikzcd}$$

where f is an injective weak equivalence and p a Kan fibration, there is a diagonal filler: $Y \to E$.

3. Conversely, if a map $p : E \to B$ has the property in 2. with respect to injective weak equivalences, then it is a Kan fibration. It follows, that Kan fibrations are stable under composition.

4. Injective weak equivalences are stable under pullback along Kan fibrations.

Interpretation of Type Theory (still too simple-mindedly)
We now have a category (simplicial sets) which is cartesian closed, and a class of maps (the Kan fibrations) which has the desirable properties mentioned before.
Therefore, we can interpret a dependent type $x{:}A \vdash B(x)$ type as a Kan fibration $B \to A$.
The Identity type: $x, x'{:}A \vdash \mathsf{Id}_A(x, x')$ type is interpreted by the *path space* PA for A: factor the diagonal embedding $A \to A \times A$ as $A \xrightarrow{r_A} PA \xrightarrow{t_A} A \times A$, with r_A an injective weak equivalence and t_A a Kan fibration. Then r_A will be our interpretation of the term refl_A.

Now, we need to interpret the term J given in the inference rules for the identity type. So assume we have a type B dependent on Id_A; which has been interpreted as a Kan fibration: $B \xrightarrow{p} PA$. And suppose we have a term $x{:}A \vdash t(x){:}B(x, x, \mathrm{refl}_A(x))$. Then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[d, "r_A"'] & B \arrow[d, "p"] \\ PA \arrow[r, "\mathrm{id}"'] & PA \end{tikzcd}$$

and the rule says basically that we need a diagonal filler in this diagram. But such is guaranteed to exist, since r_A is an injective weak equivalence and p is a Kan fibration.

So, we are done. Or, aren't we?

In fact, there is a serious flaw in this approach: since in the type theory, we may have variables present on which the whole situation depends, and we can carry out substitutions for these variables, we actually need a *coherent choice* of liftings (diagonal fillers).

But such a coherent system of liftings is not known to exist!

Solution (Voevodsky)

Fix a strongly inaccessible cardinal κ and consider only κ-small Kan fibrations.

There is a *universal* such: a Kan fibration

$$\begin{array}{c} \widetilde{\mathcal{U}} \\ \Big\downarrow{\scriptstyle p_{\mathcal{U}}} \\ \mathcal{U} \end{array}$$

such that every κ-small Kan fibration is a pullback of $p_{\mathcal{U}}$.

Now, consider the pullback

$$\begin{tikzcd}
\widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \arrow[r] \arrow[d] & \widetilde{\mathcal{U}} \arrow[d, "p_{\mathcal{U}}"] \\
\widetilde{\mathcal{U}} \arrow[r, "p_{\mathcal{U}}"'] & \mathcal{U}
\end{tikzcd}$$

and the canonical arrow $\widetilde{\mathcal{U}} \to \widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}}$.

Factor this map as

$$\widetilde{\mathcal{U}} \xrightarrow{r_{\mathcal{U}}} \mathrm{Eq}(\mathcal{U}) \xrightarrow{E_{\mathcal{U}}} \widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}}$$

where $r_{\mathcal{U}}$ is an injective weak equivalence and $E_{\mathcal{U}}$ is a Kan fibration.

Now suppose we have a dependent type, interpreted as a small Kan fibration $B \to A$.

We have a pullback diagram

$$
\begin{tikzcd}
B \times_A B \arrow[r] \arrow[d] & \widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}} \arrow[d] \\
B \arrow[r] & \widetilde{\mathcal{U}}
\end{tikzcd}
$$

Define the Identity type over $B \times_A B$, Id_B, to be the pullback of the arrow

$$E_{\mathcal{U}} : \mathrm{Eq}(\mathcal{U}) \to \widetilde{\mathcal{U}} \times_{\mathcal{U}} \widetilde{\mathcal{U}}$$

The map $\mathrm{refl}_B : B \to \mathrm{Id}_B$ is the pullback of $r_{\mathcal{U}} : \widetilde{\mathcal{U}} \to \mathrm{Eq}(\mathcal{U})$. This is not necessarily an injective weak equivalence!

From the generic small Kan fibration $\widetilde{\mathcal{U}} \xrightarrow{p_{\mathcal{U}}} \mathcal{U}$ we can construct a generic diagram

$$(GD) \qquad \begin{array}{ccc} \mathcal{A} & \xrightarrow{\delta} & \mathcal{C} \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \mathcal{E} & \xrightarrow[\mathrm{id}]{} & \mathcal{E} \end{array}$$

with two properties:

1. ρ is an injective weak equivalence and γ is a Kan fibration;

2, Every diagram of interpreted types:

$$(TD) \qquad \begin{array}{ccc} A & \xrightarrow{d} & C \\ {\scriptstyle r_A}\downarrow & & \downarrow{\scriptstyle c} \\ \mathrm{Id}_A & \xrightarrow[\mathrm{id}]{} & \mathrm{Id}_A \end{array}$$

is a pullback of (GD). Since we have a diagonal filler in (GD), we have a coherent choice of diagonal fillers for all diagrams (TD).

Voevodsky's aim is to base a new foundation of (constructive) mathematics on the following stratification of (homotopy types of) spaces:
A space is of *h*-level 0 if it is contractible
A space is of *h*-level $n+1$ if for any two points x, y, the space of paths from x to y is of *h*-level n.

0. There is exactly one homotopy type of *h*-level 0: the one-point space *pt*
1. There are exactly two homotopy types of *h*-level 1: *pt* and the empty space. Hence, level 1 is the set of *Boolean truth values*.
2. Spaces of level 2 are sums of contractible components; that is, *sets*.
3. Spaces of level 3 are *groupoids*.

$\vdots$

If the intuition of types as spaces is good, we should be able to do some basic homotopy theory inside Type Theory, with the Identity types for path spaces.

For a type A and a term $a{:}A$ define

$$
\begin{array}{ll}
\pi_1(A,a) = \mathrm{Id}_A(a,a) & a_1 = \mathrm{refl}_A(a){:}\pi_1(A,a) \\
\pi_2(A,a) = \pi_1(\pi_1(A,a),a_1) & a_2 = \mathrm{refl}_{\pi_1(A,a)}(a_1){:}\pi_2(A,a) \\
\vdots &
\end{array}
$$

One can prove: $\pi_n(A,a)$ is a group, and it is abelian for $n \geq 2$.

Suppose A is a type, $a{:}A$ a term, and t is a term of type $\prod x{:}A.\mathrm{Id}_A(a,x)$
(t witnesses that "A is path connected")
Suppose $x{:}A \vdash B(x)$ type, which we think of as a fibration

$$\sum x{:}A.B(x) \to A$$

We should have a long exact sequence

$$\begin{aligned}&\pi_n(B(a),b) \to \pi_n(\textstyle\sum x{:}A.B(x),\langle a,b\rangle) \to \\ &\pi_n(A,a) \to \pi_{n-1}(B(a),b) \to \cdots \\ &\qquad\qquad \cdots \to \pi_0(\textstyle\sum x{:}A.B(x),\langle a,b\rangle) \to \pi_0(A,a) = 1\end{aligned}$$

This is indeed the case (Voevodsky, Coquand)