Partial Combinatory Algebras – Variations on a Topos-theoretic Theme

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UGC Seminar, April 6, 2021

Computability Theory

A function $f : U \to \mathbb{N}$ for $U \subseteq \mathbb{N}$ is called a *partial function* on $\mathbb{N}$; note that a partial function may be total (in case $U = \mathbb{N}$).
Such a function is *computable* if there is a Turing machine T such that for all $n \in \mathbb{N}$ we have:

- if $n \in U$ then T, with input n, reaches a halting state and outputs $f(n)$;
- if T, with input n, reaches a halting state then $n \in U$.

Think of a Turing machine as a program in a very primitive computer language.

We can enumerate all Turing machines: $T_1, T_2, \ldots$. To every T_i corresponds a partial function ϕ_i as before; the domain of ϕ_i is the set

$$\{n \in \mathbb{N} \mid T \text{ reaches a halting state with input } n\}$$

We may write nm for $\phi_n(m)$. This is not associative; when we write $n_1 n_2 \cdots n_k$ we mean $(\cdots((n_1 n_2)n_3)\cdots)n_k$.
Since the ϕ_i are *partial* functions, such expressions need not denote anything. We write: $nm\downarrow$ to indicate that m is in the domain of ϕ_n.

Subsets of $\mathbb{N}^k$ are, in Computability theory, often called 'problems'; the 'problem' is to decide by an algorithm whether or not a given k-tuple of natural numbers is an element of that set. The algorithm (which we identify with a Turing machine) is then a 'solution' of the problem. One of the oldest such problems was the *Halting Problem* (Turing): the set

$$H = \{(n, m) \mid m \text{ is in the domain of } \phi_n\}$$

And Turing proved:
Theorem The halting problem is unsolvable (i.e., has no solution). One also says that H is an *undecidable set*.

The theory of computability aims to classify subsets of $\mathbb{N}^k$ in terms of 'difficulty to calculate'. An important tool is the notion of *Turing reducibility*: for subsets A, B of $\mathbb{N}^k$, the notation $A \leq_T B$ (A is Turing reducible to B) if a Turing machine can decide the question '$n \in A$?' provided it has access to answers to '$m \in B$?' (for example, by consulting a database for B). Turing said the machine may 'consult an oracle'.

Examples of similar structures:
$\mathcal{K}_2$ ("Kleene's second model") is the set $\mathbb{N}^{\mathbb{N}}$ of functions from $\mathbb{N}$ to $\mathbb{N}$. We assume a coding of sequences $\langle a_0, \ldots a_{n-1} \rangle$. For functions α, β, we let $\alpha\beta\downarrow$ if and only if for each natural number n there is some k such that

$$\alpha(\langle n, \beta(0), \ldots, \beta(k-1) \rangle) > 0$$

and we let $\alpha\beta(n) = \alpha(\langle n, \beta(0), \ldots, \beta(k-1) \rangle) - 1$ for the least such k.
The domain of the partial function ϕ_α is always a G_δ-set; ϕ_α is continuous on its domain.

Examples of similar structures (continued)
A *total* structure of this kind was defined by Dana Scott: let $\mathcal{S}$ be the powerset of $\mathbb{N}$. We assume bijections:

$$\begin{aligned} \langle \cdot, \cdot \rangle : &\quad \mathbb{N}^2 \to \mathbb{N} \\ e_{-} : &\quad \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathbb{N}) \end{aligned}$$

Let $AB = \{y |\, \text{for some } n, e_n \subseteq B \text{ and } \langle n, y \rangle \in A\}$ The functions ϕ_A are continuous when $\mathcal{S}$ is given the *Scott topology*: identify $\mathcal{S}$ with the set of all functions $\mathbb{N} \to \{0, 1\}$; give $\{0, 1\}$ the Sierpinski topology (with $\{1\}$ the one nontrivial open set) and $\mathcal{S}$ the product topology).

There is a common axiomatics underlying these structures; we speak of *Partial Combinatory Algebras* (PCAs)
Peter Johnstone therefore calls PCAs “Schönfinkel algebras”.
Which prompts the following short biographical intermezzo:

Moses Ilyich (or is it *Isayevich*?) *Schönfinkel* is one of the more mysterious figures in the history of logic. He was born in 1889 (or was it 1887?) in Ukraina. He worked from 1914 (!) to 1924 under Hilbert in Göttingen, during which period one paper appeared: *Über die Bausteine der mathematischen Logik* in *Mathematische Annalen* 92, 1924. However, this paper appears to have been written by someone else, who took notes during lectures by Schönfinkel.

A second paper, coauthored by Bernays, appeared in 1927; by this time, however, Schönfinkel was already in a mental hospital in Moscow.

He died in 1942 in Moscow; his papers were used for firewood by his neighbours.

Stephen Wolfram, who has a voluminous piece about Schönfinkel on his web page, also relates that his mother was from a family called "Lurie"; and the Lurie's were business partners of father Schönfinkel.

A Partial Combinatory Algebra is a set A with a partial binary operation $(a, b) \mapsto ab$ and special elements k and s, which satisfy:

$$\begin{gathered} \mathsf{k}x \downarrow \\ (\mathsf{k}a)b = a \\ (\mathsf{s}a)b \downarrow \end{gathered}$$

and: if $ac(bc) \downarrow$ then $\mathsf{s}abc \downarrow$ and

$$\mathsf{s}abc = (ac)(bc)$$

.

The letter k stands for "Konstante Funktion"; the letter s is mysteriously called "Verschmelzungsfunktion" (blending function). The original (Schönfinkel's) aim: to provide an alternative foundation of mathematics in which not sets, but *functions* are the primitive notion.

We use the following conventions for brackets and other notations: a statement $t = s$ implies that t, s and all their subterms are defined.
We write $t \preceq s$ to mean: if $s\downarrow$ then $t = s$. We write $t \simeq s$ to mean $t \preceq s$ and $s \preceq t$.
Examples: $\mathsf{s}abc \preceq ac(bc)$; $\mathsf{k}(bx) \simeq bx$.

Basic facts about PCAs

Let A be a PCA.

We consider expressions obtained from variables $(x, y, z, u, v, \ldots)$, elements of A $(a, b, c, \ldots)$, and the juxtaposition operation: e.g., x, a, $x(ab)y$, $xayb$.

For any such expression t in variables $x_0, \ldots, x_n$ there is an element $\Lambda x_0 \cdots x_n.t$ with the following properties: for each tuple $a_0, \ldots, a_n$ from A we have

- $(\Lambda x_0 \cdots x_n.t)a_0 \cdots a_{n-1} \downarrow$
- $(\Lambda x_0 \cdots x_n.t)a_0 \cdots a_n \preceq t(a_0, \ldots, a_n)$

For example: for $\Lambda x.x$ one can take skk: $\mathsf{skk}a = \mathsf{k}a(\mathsf{k}a) = a$.

Let $\mathsf{p} = \Lambda xyz.zxy$ so $\mathsf{p}ab = \Lambda z.zab$; let $\mathsf{p}_0 = \Lambda v.v\mathsf{k}$ and let $\mathsf{p}_1 = \Lambda v.v(\Lambda wu.u)$. Then $\mathsf{p}_0(\mathsf{p}ab) = a$ and $\mathsf{p}_1(\mathsf{p}ab) = b$ so p is an *ordered pair* operator, with *unpairings* p_0 and p_1.

There are also *Booleans* t and f and a *definition by cases* term C satisfying $C\mathsf{t}ab = a$ and $C\mathsf{f}ab = b$.

Some Computability theory in a PCA A

There is a copy of $\mathbb{N}$ in A: $\{\bar{n}\,|\,n\in\mathbb{N}\}$, the *Curry numerals*.

For every k-ary partial recursive function ϕ there is an element a_ϕ of A simulating ϕ: for all $n_1,\ldots,n_k\in\mathbb{N}$,

$$a_\phi\bar{n}_1\cdots\bar{n}_k\preceq\overline{\phi(n_1,\ldots,n_k)}$$

We can manipulate finite sequences $\langle a_0,\ldots,a_{k-1}\rangle$ of elements of A. For example we have for suitable $c,d\in A$:

$$c\bar{i}\langle a_0,\ldots,a_{k-1}\rangle=a_i$$
$$d\langle a_0,\ldots,a_{k-1}\rangle=\bar{k}$$

Some Computability theory in a PCA A (continued) We have a *recursion theorem* in every PCA A: there are elements y, z satisfying, for each $f \in A$:

i) $\mathrm{y}f \preceq f(\mathrm{y}f)$

ii) $\mathrm{z}f \downarrow$

iii) $\mathrm{z}fx \preceq f(\mathrm{z}f)x$ for all $x \in A$.

Theorem. Let A be a PCA. For every computable function F on the natural numbers, there is an element ϕ of A satisfying $\phi\overline{n} \simeq \overline{F(n)}$ (here $\overline{n}$ is the Curry numeral corresponding to the natural number n).

In Andy Pitts' thesis (1981) and a paper by Hyland, Johnstone and Pitts (1980) it is explained how every PCA A gives rise to a topos, the *realizability topos* over A, $\mathrm{RT}(A)$.
Hyland's paper "The effective topos" describes the topos $\mathrm{RT}(\mathcal{K}_1)$ ($\mathcal{K}_1$ is the PCA of indices of Turing machines, out first example) in great detail.
The starting point: given a PCA A we have a category $\mathrm{Ass}(A)$ of *assemblies* over A.

An assembly over A is a pair (X, E) where X is a set and $E(x)$ is a *nonempty* subset of A, for each $x \in X$.
A *morphism of assemblies* $(X, E) \to (Y, F)$ is a function $f : X \to Y$ of sets, for which there is an element $a \in A$ such that for all $x \in X$ and all $b \in E(x)$, $ab \in F(f(x))$. One says that a *tracks* the function f.

The category Ass(A) is locally cartesian closed, regular, has a weak subobject classifier (is a quasi-topos). Moreover, Ass(A) comes with an adjunction

$$(\Gamma : \mathsf{Ass}(A) \to \mathrm{Set}) \dashv (\nabla : \mathrm{Set} \to \mathsf{Ass}(A))$$

$\Gamma(X, E) = X$; $\nabla(X) = (X, \lambda x.A)$.
The category Ass(A) also has a *natural numbers object* $N = (\mathbb{N}, E)$ with $E(n) = \{\bar{n}\}$.

Structure of Ass(A):
Product $(X, E) \times (Y, F)$ is $(X \times Y, G)$ where
$G(x, y) = \{\mathsf{p}ab \,|\, a \in E(x), b \in F(y)\}$.
Exponent $(Y, F)^{(X,E)}$ is (Z, G) where Z is the set of morphisms $(X, E) \to (Y, F)$ in Ass(A), and $G(f)$ is the set of elements a which track f.

Example. Let us consider, in $\mathrm{Ass}(\mathcal{K}_1)$, the finite type structure over the natural numbers object N. The natural numbers object is isomorphic to $(\mathbb{N}, E)$ where $E(n) = \{n\}$.
We have the basic type o and for types σ, τ the arrow type $\sigma \Rightarrow \tau$.
In $\mathrm{Ass}(\mathcal{K}_1)$ we form objects X_σ for each type σ, starting with $X_o = N$ and taking exponents for the arrow types.
We obtain the structure of "hereditarily effective operations" of Kreisel-Troelstra; one of the models of the system HA^ω of intuitionistic arithmetic in all finite types. This was Hyland's original motivation for developing the effective topos.

The realizability topos $\mathrm{RT}(A)$ is the *exact completion* of the *regular category* $\mathrm{Ass}(A)$. One formally adds quotients of equivalence relations. Details are skipped.

The category $\mathrm{Ass}(A)$ is a full subcategory of $\mathrm{RT}(A)$. Actually, the category Set is the category of $\neg\neg$-sheaves in $\mathrm{RT}(A)$, and $\mathrm{Ass}(A)$ is the category of $\neg\neg$-separated objects (the objects X for which the statement $\forall xy \in X(\neg\neg(x = y) \rightarrow x = y)$ holds).

We now wish to understand: how functorial is the construction $A \mapsto \mathrm{RT}(A)$?

It turns out that there is a very nice categorical structure on the class of PCAs, which was first explored by John Longley in his thesis (1995). It has the following features:

It ties up with the standard notion of morphism for toposes, namely: geometric morphisms (Johnstone 2013, Faber/vO 2014).

It ties up with standard notions of classical recursion theory (Longley 1995, vO 2006, Longley/Normann 2015, Faber/vO 2016).

Applicative morphisms of PCAs

Let A, B be PCAs. An *applicative morphism* $A \to B$ is a total relation γ (we think of γ as a function from A to the set of nonempty subsets of B, so (A, γ) is an assembly over B) for which there is an element $\mathsf{r} \in B$ which satisfies:

For each pair a, a' of elements of A and $b \in \gamma(a), b' \in \gamma(a')$, if $aa' \downarrow$ in A then $\mathsf{r}bb' \downarrow$ in B, and $\mathsf{r}bb' \in \gamma(aa')$.

The element r *realizes* the morphism γ. Composition of morphisms is composition of total relations.

We think of γ as a *simulation* in B of computations in A; the element r is a machine that translates code for an A-program into code for a B-program.

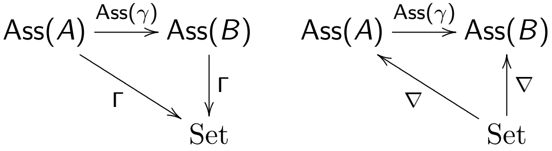
Examples of applicative morphisms

$\delta_1 : \mathcal{K}_1 \to A$: $\delta_1(n) = \{\bar{n}\}$ is the essentially unique applicative morphism $\mathcal{K}_1 \to A$ (up to a suitable notion of isomorphism of applicative morphisms)

$\delta_2 : \mathcal{K}_2^{\mathrm{rec}} \to \mathcal{K}_1$: $\delta_2(\phi) = \{e \in \mathbb{N} \,|\, \phi = \varphi_e\}$. Think of what a realizer of this morphism does; how it simulates the action of $\mathcal{K}_2^{\mathrm{rec}}$ in $\mathcal{K}_1$!

There are interesting applicative morphisms between $\mathcal{K}_2$ and $\mathcal{S}$ in both directions.

Theorem (Longley, 1995): every applicative morpjism $A \xrightarrow{\gamma} B$ gives rise to a regular functor $\mathsf{Ass}(\gamma) : \mathsf{Ass}(A) \to \mathsf{Ass}(B)$ which makes the diagrams



commute. Conversely, every regular functor making the two diagrams commute, is of the form $\mathsf{Ass}(\gamma)$ for some applicative morphism $\gamma : A \to B$.

A *geometric morphism* of toposes $f : \mathcal{F} \to \mathcal{E}$ consists of an adjoint pair

$$(f^* : \mathcal{E} \to \mathcal{F}) \dashv (f_* : \mathcal{F} \to \mathcal{E})$$

such that the left adjoint f^* preserves finite limits.
Examples: 1. If $\mathcal{F}$ and $\mathcal{E}$ are categories of sheaves over sober spaces X and Y, respectively, then these correspond exactly to continuous maps $X \to Y$.
2. The adjunction $\Gamma \dashv \nabla$ between Set and Ass(A) extends to a geometric morphism $\mathrm{Set} \to$ RT(A), which embeds Set as the category of $\neg\neg$-sheaves in RT(A)
What do geometric morphisms between realizability toposes look like?

Fundamental observation by P.T. Johnstone: Every geometric morphism $\mathrm{RT}(A) \to \mathrm{RT}(B)$ restricts to an adjunction between the categories of assemblies.
The left adjoint of such a restriction is always a regular functor commuting with the Γ's and ∇'s, and therefore corresponds to an applicative morphism $B \xrightarrow{\gamma} A$. The question then is:
For which applicative morphisms $\gamma : B \to A$ does the regular functor $\mathrm{Ass}(\gamma) : \mathrm{Ass}(B) \to \mathrm{Ass}(A)$ have a right adjoint?

Answer: (Hofstra/vO 2003; Johnstone 2013) For an applicative morphism $\gamma : B \to A$ the functor $\mathrm{Ass}(\gamma)$ has a right adjoint if and only if γ satisfies the following condition:

> There is an element $q \in A$ such that for each $a \in A$ there exists a $b \in B$ satisfying $q\gamma(b) = \{a\}$

Here $q\gamma(b) = \{a\}$ means: for all $a' \in \gamma(b)$, $qa' = a$.

Special case of geometric morphisms: inclusions

A geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is called an *inclusion* if the right adjoint f_* is full and faithful. In the case of categories of sheaves over spaces, this corresponds to an *embedding* of topological spaces.

Here I wish to draw attention to some specific inclusions between realizability toposes.

Definition: Let A and B be PCAs; let us write $\mathsf{t}_A, \mathsf{f}_A$ for the Booleans in A and ditto $\mathsf{t}_B, \mathsf{f}_B$ for B.

An applicative morphism $\gamma : A \to B$ is *decidable* if there is an element $d \in B$ such that $d\gamma(\mathsf{t}_A) = \{\mathsf{t}_B\}$ and $d\gamma(\mathsf{f}_A) = \{\mathsf{f}_B\}$.

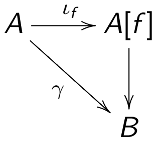
Equivalently, the functor $\mathsf{Ass}(\gamma)$ preserves finite sums.

Note, that if $\mathsf{Ass}(\gamma)$ has a right adjoint, γ is necessarily decidable.

Computations in PCAs with an oracle

Let $\gamma : A \to B$ be an applicative morphism. A partial function $f : A \rightharpoonup A$ is *representable* w.r.t. γ if there is an element $b \in B$ satisfying: for each $a \in A$, if $f(a)\downarrow$ then $b\gamma(a) \subseteq \gamma(f(a))$.

Theorem (vO 2006): Given PCA A and partial function f on A, there is a PCA $A[f]$ which is universal with the property that there is a decidable applicative morphism $\iota_f : A \to A[f]$ w.r.t which f is representable: if $\gamma : A \to B$ is decidable and f is representable w.r.t. γ, then γ factors uniquely through ι_f:



Applying this construction to $\mathcal{K}_1$ gives us the PCA of "computations with oracle f".

Note, that this construction gives us a notion of "Turing reducibility in A": if f and g are partial functions on A, then $f \leq_T g$ if and only if f is representable w.r.t. $\iota_g : A \to A[g]$. Equivalently: for every decidable applicative morphism $A \stackrel{\gamma}{\to} B$ we have: if g is representable w.r.t. γ, then so is f.

An extension of the "oracle" result (Faber/vO 2016)
Given a PCA A, we can define what we call an "effective operation of type 2" in A, and we have, for any partial function $F : A^A \rightharpoonup A$ a similar universal solution for "forcing F to be an effective operation": a decidable applicative morphism $\iota_F : A \to a[F]$ with the expected universal property.
We have the following result (which should not come unexpected): For the Kleene functional E ($E(f) = 0$ if and only if $\exists n f(n) = 0$) we have: a function $\mathbb{N} \to \mathbb{N}$ is representable w.r.t. $\mathcal{K}_1[E]$ if and only if the function f is hyperarithmetical.
This opens up the possibility of "realizability with hyperarithmetical functions"; this is a sheaf subtopos of the effective topos in which there is a model of Peano Arithmetic (with classical logic!). Such a model cannot exist in the effective topos.

In a recent paper, Jetze Zoethout takes this one step further. He explains why a straightforward extension to "third-order functionals" is not to be expected; however, employing a "lax" version of PCAs (the equations hold "up to inequality") one can obtain, for such a PCA A and third-order Φ, a PCA $A[\Phi]$ enjoying a weaker universal property.