Retake Exam Category Theory and Topos Theory
June 16, 2014; 10:00–13:00

THIS EXAM CONSISTS OF FIVE PROBLEMS
Advice: first do those problems you can do right away; then, start thinking about the others.
Good luck!

Exercise 1.

a) Show by a counterexample that a faithful functor need not reflect terminal objects.

b) Let $\mathcal{C}$ be a category with equalizers. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves equalizers and reflects isomorphisms. Prove that F is faithful.

Exercise 2. Recall that an object P of a category is *projective* if for every diagram

$$\begin{array}{ccc} & & P \\ & & \downarrow{\scriptstyle g} \\ A & \xrightarrow[e]{} & B \end{array}$$

with e epi, there is an arrow $f : P \to A$ such that $ef = g$. A category is said to *have enough projectives* if for every object X there is an epimorphism $P \to X$ with P projective.

Let $\mathcal{C} \underset{G}{\overset{F}{\leftrightarrows}} \mathcal{D}$ be an adjunction ($F \dashv G$). Suppose that $\mathcal{D}$ has enough projectives and that F preserves projectives. Prove that G preserves epimorphisms.

Exercise 3. Let $\mathcal{C}$ be a category with finite coproducts. For a fixed object A of $\mathcal{C}$ we consider the category $A/\mathcal{C}$: an object of $A/\mathcal{C}$ is an arrow $A \to X$ in $\mathcal{C}$, and a morphism from $A \stackrel{f}{\to} X$ to $A \stackrel{g}{\to} Y$ is an arrow $X \stackrel{\alpha}{\to} Y$ in $\mathcal{C}$ satisfying $\alpha f = g$.

a) Consider the forgetful functor: $U : A/\mathcal{C} \to \mathcal{C}$ which sends $A \to X$ to X (and morphisms to themselves). Determine whether U has a left and/or a right adjoint.

b) Is the functor U of part a) monadic? Motivate your answer.

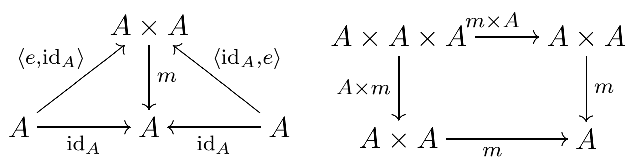
Exercise 4. Let $\mathcal{E}$ be a topos with subobject classifier $1 \xrightarrow{t} \Omega$. Consider the subobject $1 \xrightarrow{\langle t,t \rangle} \Omega \times \Omega$ of $\Omega \times \Omega$, and its classifying map $F : \Omega \times \Omega \to \Omega$.

a) Suppose A and B are subobjects of an object X, classified by maps $\phi, \psi : X \to \Omega$ respectively. What is the subobject of X classified by the composite map

$$X \xrightarrow{\langle \phi,\psi \rangle} \Omega \times \Omega \xrightarrow{F} \Omega \ ?$$

b) Let $P \xrightarrow{p} \Omega \times \Omega$ be the equalizer of F and the first projection. Given two subobjects A and B of an object X, classified by ϕ and ψ as before, show that the map $\langle \phi, \psi \rangle : X \to \Omega \times \Omega$ factors through P if and only if $A \leq B$.

Exercise 5. In a category with finite products $\mathcal{E}$, a *monoid object* is an object A together with maps $1 \xrightarrow{e} A$ and $A \times A \xrightarrow{m} A$ such that the diagrams



commute. We have a category $\mathrm{Mon}(\mathcal{E})$ of monoid objects (and monoid maps) in $\mathcal{E}$ and a forgetful functor $\mathrm{Mon}(\mathcal{E}) \to \mathcal{E}$. The category $\mathcal{E}$ is said to *have free monoids* if this forgetful functor has a left adjoint.

a) Prove that for every small category $\mathcal{C}$, $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ has free monoids.

b) Give an example of a small category $\mathcal{C}$ and a Grothendieck topology on $\mathcal{C}$ for which the free monoid construction of a) does not always yield a sheaf, even if we start out with a sheaf.