Exam Computability Theory

March 22, 2007, 14.00–17.00

This exam consists of 5 exercises; see also the back side
Advice: first do those exercises you can do straight away; then start thinking about the others.
SUCCES!

Exercise :
Prove that the following functions are primitive recursive:

1) $F(x) = x^x$

2) $F(x) = \underbrace{x^{x^{\cdot^{\cdot^{x}}}}}_{x \text{ times}}$

Hint: in both cases, consider an auxiliary function of more than one variable.

Exercise 2:
Call a property P of indices of total recursive functions decidable if there is a partial recursive function ϕ, such that for every index e of a total recursive function,

$$\phi(e) \;=\begin{cases} 0 & \text{if } e \text{ has property } P \\ 1 & \text{else} \end{cases}$$

Which of the following properties are decidable for indices of total recursive functions? Motivate your answer.

1) $P \;=\; \{e \,|\, \forall xy(x \leq y \rightarrow \phi_e(x) \leq \phi_e(y))\}$

2) $P \;=\; \{e \,|\, \exists x \forall y(y > x \rightarrow \phi_e(x) \leq \phi_e(y))\}$

Exercise 3:
Let R be a c.e. set. Show that the set

$$\bigcup_{x \in R} W_x$$

is also c.e.

Exercise 4:
Classify the following subsets of $\mathbb{N}$ in the arithmetical hierarchy:

a) $A = \{x \mid W_x \text{ has at most two elements}\}$

b) $B = \{x \mid \forall y \in W_x (W_y \text{ is a singleton})\}$

Prove for your classification of a) that it is complete, i.e. that A is m-complete in its class.

Exercise 5:

Show that there is an index e such that

$$W_e = \{x \mid \phi_x(e)\downarrow\}$$