Exam Computability Theory

May 27, 2013, 14.15–17.15
With Solutions

This exam consists of 5 exercises; see also the back side. Every exercise is worth 10 points; in case the exercise is divided into parts, it is indicated at each part how much this part is worth.
Advice: first do those exercises you can do straight away; then start thinking about the others.
GOOD LUCK!

Exercise 1:
Prove that the following functions are primitive recursive:

a) (5) $F(x) = x \underbrace{! \cdots !}_{x \text{ times}}$

b) (5) The function F which is such that $F(x) = 0$ for $x \leq 1$, and for $x > 1$, $F(x)$ is the cardinality of the set $\{y \mid 1 \leq y < x,\ \gcd(y, x) = 1\}$

Solution: a) Define the function $G(y, x)$ by $G(0, x) = x$ and $G(y + 1, x) = G(y, x)!$. Since the function $x \mapsto x!$ is primitive recursive, G is primitive recursive (being defined by primitive recursion from the factorial function). Now $F(x) = G(x, x)$, so F is defined by composition from G.
b) Define $G(y, x)$ as follows: $G(0, x) = 0$ and

$$G(y+1, x) = \begin{cases} 0 & \text{if } y = 0 \\ G(y, x) & \text{if } y > 0 \text{ and } \gcd(y+1, x) > 1 \\ G(y, x) + 1 & \text{otherwise} \end{cases}$$

Then since gcd is primitive recursive and G is defined by primitive recursion from gcd and definition by cases from primitive recursive relations, G is primitive recursive. Now $F(x) = G(x, x)$.

Exercise 2:
Prove that the following problems are unsolvable with respect to indices of total functions:

a) (5) $\exists x\, |e \cdot x - e \cdot (x + 1)|$ is even

b) (5) $e \cdot e \in \mathcal{K}$ (where $\mathcal{K}$ is the standard set)

Solution: a) Suppose f were an index such that for all e with ϕ_e total, $f{\cdot}e$ is defined and $f{\cdot}e = 0$ if and only if $\exists x\,|e{\cdot}x - e{\cdot}(x+1)|$ is even. By the Recursion Theorem, there is an index a satisfying for all x:

$$a{\cdot}x \;\simeq\; \begin{cases} x & \text{if for no } k \leq x,\, T(1,f,a,k) \text{ and } U(k) > 0 \\ 2 & \text{otherwise} \end{cases}$$

Then ϕ_a is total, hence $f{\cdot}a$ is defined; but if $f{\cdot}a = 0$ then $a{\cdot}x = x$ for all x so $|a{\cdot}x - a{\cdot}(x+1)|$ is never even. And if $f{\cdot}a > 0$ then from some point on, ϕ_a is constant with value 2 so then for all x large enough, $|a{\cdot}x - a{\cdot}(x+1)|$ is even. This is a contradiction with the assumption on f.
b): there is a primitive recursive function F such that for all e, $\phi_{F(e)}$ is total and $F(e){\cdot}F(e) = e$. Namely, choose an index f such that $f{\cdot}(e,x) = e$ for all e, x and let $F(e) = S^1_1(f,e)$. Now clearly $F(e){\cdot}F(e) \in \mathcal{K}$ if and only if $e \in \mathcal{K}$ so if the property in b) were decidable for indices of total functions, then $\mathcal{K}$ would be recursive; quod non.

Exercise 3:
We consider the set $A = \{e \,|\, W_e \text{ has at most } e \text{ elements}\}$.

a) (5) Show that A is a Π_1-set.

b) (5) Show that A is not a Σ_1-set, by reducing $\mathbb{N} - \mathcal{K}$ to A.

Solution: a) $e \in A$ holds if and only if for all σ: if $\mathrm{lh}(\sigma) = e+1$ and $\forall i \leq e\, T(1, e, j_1((\sigma)_i), j_2((\sigma)_i))$, then $\exists i,k \leq e\,(i \neq k \wedge j_1((\sigma)_i) = j_1((\sigma)_k))$. Therefore $A = \{e \,|\, \forall \sigma R(e,\sigma)\}$ for some primitive recursive relation R; so A is a Π_1-set.
b): Let, by the Recursion Theorem, f be an index such that

$$f{\cdot}(e,x) \;\simeq\; \begin{cases} 0 & \text{if } x < S^1_1(f,e) \\ e{\cdot}e & \text{otherwise} \end{cases}$$

Then for $F(e) = S^1_1(f,e)$ we have: $W_{F(e)}$ has at most $F(e)$ elements, if and only if $e{\cdot}e$ is undefined, so if and only if $e \in \mathbb{N} - \mathcal{K}$. We see that $(\mathbb{N} - \mathcal{K}) \leq_m A$. Since $\mathbb{N} - \mathcal{K}$ is m-complete in Π_1, A is.

Exercise 4:
We consider the set $A = \{e \,|\, \phi_e \text{ is total and eventually constant}\}$. Here, '$F$ is eventually constant' means: for some n, $\forall k, l > n\,(F(k) = F(l))$.

a) (3) Show that A is not recursively enumerable.

b) (4) Classify A in the arithmetical hierarchy: give a number n (as small as you can) and show that A is Σ_n, Π_n or Δ_n.

c) (3) Show that your classification is best possible.

Solution: a) Considering that A is extensional for indices of partial recursive functions, this is more or less immediate from the Myhill-Shepherdson Theorem.
b) A can be written as the intersection of two sets A_1 and A_2:

$$\begin{aligned} A_1 &= \{e \mid \forall x \exists y\, T(1,e,x,y)\} \\ A_2 &= \{e \mid \exists n \forall klmr\, (T(1,e,k,l) \wedge T(1,e,m,r) \wedge k,m > n) \to U(l) = U(r)\} \end{aligned}$$

Now A_1 is Π_2 and A_2 is Σ_2, so both sets are in Δ_3. Because Δ_3 is closed under intersections, A is a Δ_3-set.
c) In order to show that A is not in Π_2, take a set B which is m-complete in Σ_2 and prove $B \leq_m A$. For example, $B = \{e \mid \mathrm{rge}(\phi_e) \text{ is finite}\}$. Let f be an index such that

$$f{\cdot}(e,x) \;\simeq\; \max\{U(k) \mid k \leq x \text{ and } \exists w \leq x T(1,e,w,k)\}$$

Then for $F(e) = S^1_1(f,e)$, $\phi_{F(e)}$ is total (so $F(e) \in A_1$), and $F(e) \in A_2$ if and only if $\mathrm{rge}(\phi_e)$ is finite. Hence F reduces B to A, as desired.
In order to show that A is not in Σ_2, take a set B which is m-complete in Π_2 and prove $B \leq_m A$. Here we can take $B = \{e \mid \phi_e \text{ is total}\}$. Let f be such that

$$f{\cdot}(e,x) \;\simeq\; \begin{cases} 0 & \text{if } e{\cdot}x \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then for $F(e) = S^1_1(f,e)$, $F(e)$ is always in A_2, and $F(e) \in A_1$ if and only if ϕ_e is total (which is, if $e \in B$). So again F reduces B to A.

Exercise 5:
We consider every function $F : \mathbb{N} \to \mathbb{N}$ as a directed graph with set of points $\mathbb{N}$: for $n, m \in \mathbb{N}$, we have an edge from n to m if and only if $F(j(n,m)) = 0$. In such a graph, a *path* from m to n is a sequence of edges, such that each edge starts where the previous one ends, and the first one starts at m and the last one ends at n. A graph is *connected* if for each n and m, there is either a path from m to n or a path from n to m.

a) (5) Show that the set $\{F \mid F, \text{ considered as a graph, is connected}\}$ is a Π^0_2-subset of $\mathbb{N}^{\mathbb{N}}$

b) (5) Show that the set $\{F \mid \text{there is no infinite path in } F\}$ is not arithmetical.

Solution: a) The given set consists of those F satisfying: for all m, n, either there is σ with $(\sigma)_0 = m$ and $(\sigma)_{\mathrm{lh}(\sigma)-1} = n$ and $\forall i < \mathrm{lh}(\sigma) - 1\, F(j((\sigma)_i, (\sigma)_{i+1})) = 0$, or there is such a σ in the other direction (interchange the roles of m and n). So this is of the form $\forall nm\exists\sigma[\cdots]$ with $[\cdots]$ recursive. Hence the set is Π^0_2.
b) This should have reminded you of the set WfRec of indices of characteristic functions of recursive well-founded trees. There is a primitive recursive function F such that for all x,

$$F(x){\cdot}j(\sigma,\tau) \simeq \begin{cases} 0 & \text{if } x{\cdot}\sigma = x{\cdot}\tau = 0 \text{ and} \\ & \mathrm{lh}(\tau) = \mathrm{lh}(\sigma) + 1 \text{ and } \forall i < \mathrm{lh}(\sigma)\, (\sigma)_i = (\tau)_i \\ 1 & \text{if } x{\cdot}\sigma \text{ and } x{\cdot}\tau \text{ are defined but the condition} \\ & \text{above does not hold} \\ \text{undefined} & \text{if } x{\cdot}\sigma \text{ or } x{\cdot}\tau \text{ is undefined} \end{cases}$$

Then if ϕ_x is the characteristic function of a recursive tree, $\phi_{F(x)}$ is total, and $\phi_{F(x)}$, considered as a graph, has no infinite path precisely when ϕ_x is the characteristic function of a recursive well-founded tree. So if the set $A = \{F \mid \text{there is no infinite path in } F\}$ were arithmetical, then the set $\{x \mid \phi_x \in A\}$ would be an arithmetical subset of $\mathbb{N}$. However, it is m-complete in Π^1_1 by Theorem 6.3.3 in the notes.