Exam Foundations of Mathematics, february 3, 2021, 11.30-14.30

THIS EXAM CONSISTS OF 5 EXERCISES; SEE ALSO REVERSE SIDE.
All exercises have equal weight (10 points); your grade will be your total number of points, divided by 5. If an exercise consists of more than 1 part, it is indicated what each part is worth..
Advice: first solve those exercises you can do straight away; then, start thinking about the others. Good luck!

Exercise 1.In this exercise we consider the power set $\mathcal{P}(\mathbb{N})$ of the set $\mathbb{N}$ of natural numbers; on the set $\mathcal{P}(\mathbb{N})$ we have the operation

$$U, V \mapsto U + V = (U \cup V) - (U \cap V)$$

(see pp. 38–39 of the book).

a) (5 points) Use Zorn's Lemma to prove that there is a subset $\mathcal{A}$ of $\mathcal{P}(\mathbb{N})$ which is maximal w.r.t. the following properties:

   i) for all $U, V \in \mathcal{A}$ we have $U + V \in \mathcal{A}$.

   ii) $\mathbb{N} \notin \mathcal{A}$.

b) (2 points) Let $\mathcal{A}$ be as in part a). Prove: $\emptyset \in \mathcal{A}$.

c) (3 points) Let $\mathcal{A}$ be as in part a). Prove: for all $W \in \mathcal{P}(\mathbb{N})$ we have: either $W \in \mathcal{A}$ or $\mathbb{N} - W \in \mathcal{A}$.

Exercise 2. In this exercise we consider a nonempty, countable well-order L which has no greatest element.

a) (4 points) Prove: there is an injective, strictly increasing function $n \mapsto b_n : \mathbb{N} \to L$ (in other words, a sequence $b_0 < b_1 < b_2 < \cdots$ in L), such that for every $t \in L$ there exists $n \in \mathbb{N}$ with $t < b_n$.

b) (3 points) Write $L_{<l}$ for $\{x \in L \,|\, x < l\}$. Prove: if for each $l \in L$ there exists an injective, increasing function $L_{<l} \to \mathbb{R}$, then there is also such a function from L to $\mathbb{R}$.

c) (3 points) Show that the statement in b) is no longer valid if we drop the assumption that L is countable.

Exercise 3. In this exercise we consider the theory of Peano Arithmetic, formulated in the language L_{rings} of rings; see Example 2.5.4 (p. 59) of the book. For every natural number n there is the L_{rings}-term

$$\overline{n} \;=\; \underbrace{1+(1+(\cdots+1))}_{\text{n}}$$

Use the Compactness Theorem in order to show that there exists an L_{rings}-structure M with the following properties:

i) $\mathbb{N}$ is an elementary substructure of$M$.

ii) In M there is an element c such that for every prime number p the L_{rings}-sentence $\exists x(x{\cdot}\overline{p}=c)$ is true in M.

Exercise 4. An L-theory is said to be *model complete* if for every pair M, N of models of T the following holds: if M is a substructure of N then M is an elementary substructure of N. We consider the following property of an L-theory T:

(*) For every L-formula $\phi(\vec{u})$ there is an L-formula $\exists x_1\cdots\exists x_k\psi(\vec{x},\vec{u})$, with ψ quantifier-free, such that

$$T \models \forall\vec{u}(\phi(\vec{u}) \leftrightarrow \exists\vec{x}\psi(\vec{x},\vec{u}))$$

a) (4 points) Show: if T has the property (*) then for every L-formula ϕ there also exists an L-formula $\forall x_1\cdots\forall x_k\chi(\vec{x},\vec{u})$, with χ quantifier-free, such that

$$T \models \forall\vec{u}(\phi(\vec{u}) \leftrightarrow \forall\vec{x}\chi(\vec{x},\vec{u}))$$

b) (6 points) Show: if T has the property (*) then T is model complete.

Exercise 5. In chapter 1 of the book we defined the notion "the cardinality of set X" (notation: $|X|$) only in expressions like $|X| \leq |Y|$, with meaning: there is a 1-1 function $X \to Y$. In chapter 4 (p. 111) we define the cardinality of X as the least ordinal number α such that there exists a bijection $X \to \alpha$; let us use the notaion $\kappa(X)$ for this.

a) (5 points) Let $g : X \to \alpha$ be an injective function, with α an ordinal number. Prove: $\kappa(X) \leq \alpha$.

b) (5 points) Show that the following two statements are equivalent, for sets X and Y:

i) There exists a 1-1 function $X \to Y$

ii) $\kappa(X) \leq \kappa(Y)$