Exam Model Theory, June 19, 2023, 13:00–16:00
with solutions

Exercise 1 Let L be a language, T an L-theory. Recall that we write $T_\forall$ for the set of all universal consequences of T.

Prove that for an L-sentence ϕ the following two statements are equivalent:

i) ϕ is a consequence of $T_\forall$.

ii) ϕ is true in every substructure of a model of T.

Solution. i)⇒ii): this follows at once from the observation that every sentence in $T_\forall$ is true in every substructure of a model of T. Indeed, if $\phi \in T_\forall$ is the formula $\forall\vec{x}\psi$ with ψ quantifier-free, and M is a model of T with substructure N, then for all $\vec{m} \in M$ we have $M \models \psi(\vec{m})$, so $N \models \psi(\vec{m})$. Since $N \subseteq M$, we thus have for all $\vec{n} \in N$ that $N \models \psi(\vec{n})$, so $N \models \phi$.

ii)⇒i): We claim that every model of $T_\forall$ is a substructure of a model of T. This implies the statement we have to prove, for then if ϕ is true in every substructure of a model of T, a fortiori ϕ will be true in every model of $T_\forall$, and hence be a consequence of $T_\forall$.

In oder to prove the claim, let N be a model of $T_\forall$. A model M of T which contains N as a substructure, is a model of $\mathrm{Diag}(N) \cup T$. So assume this theory is inconsistent. Then by the compactness theorem there is some quantifier-free $L(N)$-sentence $\psi(\vec{n})$ which is true in N but such that $T \models \neg\psi(\vec{n})$. Since the constants $\vec{n}$ do not appear in T, it follows that $T \models \forall\vec{x}\neg\psi(\vec{x})$; and therefore $\forall\vec{x}\neg\psi(\vec{x})$ is an element of $T_\forall$. But clearly this sentence is not true in N, so this contradicts our assumption.

Exercise 2 In this exercise we consider models of Peano Arithmetic PA. Given such a model M, an "end extension" of M is a model N such that M is an initial segment of N (i.e., for $m \in M$ and $n \in N$ we have: if $n \leq m$ then $n \in M$). An extension $M \subseteq N$ is "proper" if $M \neq N$.

Let M be a countable model of PA. Let $L(M)$ be the language of M, and let c be a new constant.

a) (3 pts) Construct an $L(M) \cup \{c\}$-theory T such that every model of T is a proper elementary extension of M.

b) (4 pts) Construct a family of types (over M) such that the following holds: if N is a model of T which omits every type in the family, then N is a proper elementary end extension of M.

c) (2 pts) Show that none of the types you constructed in part b) is isolated.

d) (1 pt) Conclude that there exists a proper elementary end extension of M.

Solution. a) Let ElDiag(M) denote the elementary diagram of a structure M. Given a countable model M of PA, consider the $L(M) \cup \{c\}$-theory

$$T \equiv \text{ElDiag}(M) \cup \{\neg(c = m) \mid m \in M\}$$

Every model N of T is an elementary extension of M since it is a model of ElDiag(M), and it is a proper extension by the axioms on c.

b) We distinguish cases. If M is the standard model of PA, then every model of T is a proper elementary extension by a) which is also an end extension because every model of PA contains the standard numbers. So we can take the empty family of types.

For M nonstandard, let for each nonstandard $m \in M$, Σ_m be the family of $L(M)$-formulas

$$\{x < m\} \cup \{\neg(x = m' \mid m' < m\}$$

(Note that if m is standard, this family is not finitely satisfiable) and let p_m be any type extending Σ_m. It is left to you to work out that if N is a model of T which omits each p_m hence each Σ_m, then N is a proper elementary end extension of M.

c) Suppose the type p_m is isolated, for some nonstandard $m \in M$; then for some $L(M)$-formula $\phi(x)$ we have

$$(*) \quad T \models \forall x(\phi(x) \to x < m) \quad \text{and} \quad T \models \forall x(\phi(x) \to \neg(x = m')$$

for each $m' < m$ in M. Applying Compactness to the first statement of $(*)$, we find that

$$\text{ElDiag}(M) \cup \{\neg(c = m_1, \ldots, \neg(c = m_k)\} \vdash \forall x(\phi(x) \to x < m)$$

for some elements $m_1, \ldots, m_k \in M$. Since c does not occur in ElDiag(M) or $\forall x(\phi(x) \to x < m)$ we have

$$\text{ElDiag}(M) \models \forall x(\phi(x) \to x < m)$$

And applying a similar reasoning to the second statement of $(*)$ we find

$$\text{ElDiag(M)} \models \forall x(\phi(x) \rightarrow \neg(x = m'))$$

for all $m' < m$. We conclude that these sentences must hold in M, but this clearly leads to the conclusion that the formula $\phi(x)$ is inconsistent with $\text{ElDiag}(M)$.

d) By the Omitting Types Theorem, a model which omits all Σ_m exists.

Exercise 3 Let L be a countable language, κ an uncountable cardinal and M an infinite, κ-saturated structure.

- a) (3 pts) Show that $|M| \geq \kappa$.
- b) (4 pts) Show that every definable subset of M is either finite or of cardinality $\geq \kappa$.
- c) (3 pts) Show that if the algebraic closure of $\emptyset$ is infinite, it is not definable.

Solution a) If $|M| < \kappa$, then the family of $L(M)$-formulas

$$\{\neg(x = a) \,|\, a \in M\}$$

is finitely satisfiable since M is infinite, and it contains $\leq \kappa$ many parameters from M; so by κ-saturation of M it should be realized in M. But that is plainly impossible.

b) If $\phi(x)$ is a formula such that the set $A = \phi(M)$ is infinite, then we can apply the same trick as in part a) and consider the family

$$\{\phi(x) \wedge \neg(x = m) \,|\, m \in A\}$$

and conclude that it should be realized in M, which again is impossible.

c) Since the language is countable, there are only countably many algebraic formulas and therefore also only countably many elements which are algebraic over $\emptyset$. In other words, $\text{acl}(\emptyset)$ is countable and it cannot be definable by b), since κ is uncountable.

Exercise 4 In this exercise we consider a complete theory T which has infinite models, with respect to which we define Morley rank. Let M be a model of T. Show that there is a type $p \in S_1(M)$ which has Morley rank ≥ 1.

Solution. We consider a type p which extends the family of formulas

$$\sigma \;=\; \{\neg(x = a) \,|\, a \in M\}$$

For any formula $\neg(x = a)$ we have the family of formulas $q_a \;=\; \{x = b \,|\, b \neq a\}$. Each formula $x = b$ is consistent and algebraic, and therefore has Morley rank 0; and since the formulas in q_a are pairwise consistent and imply $\neg(x = a)$, we see that the latter formula has Morley rank ≥ 1.

Now let $\phi(x)$ be any formula in p. Then $\phi(x)$ cannot be algebraic, for if $a_1, \dots, a_k$ are the realizers in M of ϕ, then since p is finitely satisfiable, we should have

$$M \models \exists y(\phi(y) \wedge \neg(y = a_1) \wedge \cdots \wedge \neg(y = a_k))$$

which contradicts the assumption on $a_1, \dots, a_k$. It follows that $\phi(x)$ has Morley rank ≥ 1, and therefore $\mathrm{MR}(p) \geq 1$.