Exam Topos Theory January 21, 2019; 10:00–13:00 With solutions

Exercise 1. Let \mathcal{E} be a topos, and $j : \Omega \to \Omega$ a Lawvere-Tierney topology in \mathcal{E} . By $Sh_j(\mathcal{E})$ we denote the category of *j*-sheaves.

a) (5) Let X be a *j*-sheaf, and let $X \xrightarrow{\nu_X} \tilde{X}$ be a partial map classifier in $\operatorname{Sh}_i(\mathcal{E})$. Show that for any diagram

$$\begin{array}{c} M \xrightarrow{m} Y \\ f \\ \downarrow \\ \chi \end{array}$$

in ${\mathcal E}$ with m mono, there exists an arrow $\tilde f:Y\to \tilde X$ such that the square

$$\begin{array}{ccc} & M & \stackrel{m}{\longrightarrow} Y \\ (*) & f & & & \downarrow \\ & X & & & \downarrow \\ & X & \stackrel{m}{\longrightarrow} X \end{array}$$

commutes.

b) (5) Now suppose the mono *m* represents a *j*-closed subobject of *Y*. Show that there is a *unique* $\tilde{f} : Y \to \tilde{X}$ making the square (*) a pullback.

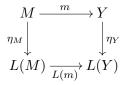
Solution: a): consider the sheafification functor L and the natural transformation $\eta : id \Rightarrow L$. We know that the functor L preserves finite limits; in particular it preserves monos. Since X is a *j*-sheaf, the partial map diagram transposes to a diagram

$$\begin{array}{c} L(M) \xrightarrow{L(m)} L(Y) \\ \bar{f} \\ \downarrow \\ X \end{array}$$

which is a partial map $L(Y) \to X$ in $\operatorname{Sh}_j(\mathcal{E})$. By the property of the partial map classifier \tilde{X} , we have a unique arrow $\tilde{f}: L(Y) \to \tilde{X}$ making the diagram

$$\begin{array}{c} L(M) \xrightarrow{L(m)} L(Y) \\ \bar{f} \downarrow \qquad \qquad \qquad \downarrow \tilde{f} \\ X \xrightarrow{\nu_X} \tilde{X} \end{array}$$

a pullback. If we compose this with the naturality square



and define \tilde{f} to be the composite $\tilde{f}\eta_Y$, we have our commuting square.

b): here one has to see that the map $M \xrightarrow{m} Y$ is *j*-closed precisely when the naturality square is a pullback. This is so because the universal closure operation corresponding to the Lawvere-Tierney topology *j* sends a mono $m: M \to Y$ to the pullback of L(m) along η_Y . Clearly now, if *m* is *j*-closed then the construction given for part a) yields a pullback square.

For uniqueness, suppose the square (*) is a pullback. Since $X \xrightarrow{\nu_X} \tilde{X}$ is a diagram of sheaves, (*) transposes to a diagram

$$L(M) \xrightarrow{L(m)} L(Y)$$

$$\downarrow \bar{f} \qquad \qquad \qquad \downarrow \bar{f}$$

$$X \xrightarrow{\nu_X} \tilde{X}$$

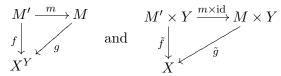
which, modulo the isomorphisms $X \simeq L(X)$ and $\tilde{X} \simeq L(\tilde{X})$, is just the *L*-image of the diagram (*), and hence a pullback; we see that \tilde{f} represents the partial map \bar{f} in $\text{Sh}_j(\mathcal{E})$, and is therefore uniquely determined by m and \bar{f} . Hence its transpose \tilde{f} is uniquely determined by m and f.

Exercise 2. Again, \mathcal{E} is a topos and j is a Lawvere-Tierney topology in \mathcal{E} . Let $\operatorname{Sep}_j(\mathcal{E})$ be the full subcategory of \mathcal{E} on the j-separated objects, and let $M : \mathcal{E} \to \operatorname{Sep}_j(\mathcal{E})$ be left adjoint to the inclusion functor $\operatorname{Sep}_j(\mathcal{E}) \to \mathcal{E}$.

- a) (4) Prove that if X is *j*-separated then so is X^Y , for any Y.
- b) (4) Prove that the functor M preserves finite products.
- c) (2) Does M preserve equalizers in general? Motivate your answer.

Solution: a): the simplest was to remark that we know this for sheaves: if X is a sheaf then X^Y is a sheaf. Now if X is *j*-separated, X is a subobject of a sheaf, say we have a mono $m: X \to Z$ with Z a sheaf. Then $m^Y: X^Y \to Z^Y$ is a monomorphism (since the functor $(-)^Y$, being a right adjoint, preserves monos) of X^Y into the sheaf Z^Y ; so X^Y is separated.

Alternatively, suppose $M' \xrightarrow{m} M$ is *j*-dense and $M' \xrightarrow{f} X^Y$ is a map. By the exponential adjunction, there is a natural bijection between commutative triangles



and, by stability of the closure operation, the map $m \times \text{id}$ is dense if m is. Since X is separated, there is at most one \tilde{g} making the right hand triangle commute. So there is at most one g making the left hand triangle commute; that is, X^Y is separated.

b): this is similar to the proof for L in the lecture notes. For the binary case one proves, for arbitrary separated objects X, that there is a natural bijective correspondence $\operatorname{Sep}_j(MY \times MZ, X) \simeq \operatorname{Sep}_j(M(Y \times Z), X)$, and applies the Yoneda lemma. It is trivial that $M(1) \simeq 1$.

c): you got full points if you remarked that if M preserved equalizers then M would preserve all finite limits, and therefore would define a subtopos of \mathcal{E} . Not every category of separated objects is itself a topos. For a simple example, take the poset **2**, the linear order with 2 elements, and consider the $\neg\neg$ -topology on Set². Note that objects of this category are arrows in Set; the category of $\neg\neg$ -separated objects is the full subcategory on the injective functions. This is not a topos.

Exercise 3. Let P be the poset of finite 01-sequences ordered by extension: $\sigma \leq \tau$ if and only if σ is an initial segment of τ . We consider the toposes Set/P (the slice topos, where P is regarded as just a set) and \widehat{P} , the category of presheaves on the poset P.

Show that there exist both a surjection and an embedding from Set/P to \widehat{P} .

Solution: consider P as a category and let P_{dis} be the discrete category on the objects of P. It is an easy observation that Set/P is equivalent to $\widehat{P_{\text{dis}}}$.

We know from the lectures that a functor $F : \mathcal{C} \to \mathcal{D}$ induces a geometric morphism $\widehat{F} : \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ which is a surjection if F is surjective on objects, and an embedding if F is full and faithful.

So it suffices to find functors $F, G : P_{\text{dis}} \to P$ such that F is surjective on objects and G is full and faithful. Note that any map from P to itself gives a functor $P_{\text{dis}} \to P$.

For F we can take the identity function; this is surjective on objects (this induces the geometric morphism that most of you found). For G, we

need a function from P to P such that *different* elements p and q are sent to *incomparable* elements G(p), G(q). Now P is a countably infinite set; choose an enumeration $(e_n)_{n \in \mathbb{N}}$ of P. Define G such that $G(e_n)$ is the sequence $\underbrace{0 \cdots 0}_{n} 1$. Then G is a full and faithful functor, and induces an embedding $\operatorname{Set}/P \simeq \widehat{P_{\text{dis}}} \to \widehat{P}$.

Exercise 4. Recall that in a topos \mathcal{E} an object X is *internally projective* if the functor $(-)^X$ preserves epimorphisms; \mathcal{E} is said to *satisfy the internal axiom of choice* (IC) if every object of \mathcal{E} is internally projective.

Show that the following two assertions are equivalent:

- i) \mathcal{E} satisfies IC and 1 is projective in \mathcal{E} .
- ii) Every object of \mathcal{E} is projective.

Solution: i) \Rightarrow ii) is most elegantly proven by observing that the functor $\mathcal{E}(X, -)$ is isomorphic to $\mathcal{E}(1, (-)^X)$, which is the composition of $(-)^X$: $\mathcal{E} \to \mathcal{E}$ and $\mathcal{E}(1, -) : \mathcal{E} \to$ Set. The first of these preserves epis because X is internally projective, and the second one does because 1 is projective. Hence $\mathcal{E}(X, -)$ preserves epis; that is, X is projective.

ii) \Rightarrow i): Let $f : A \rightarrow B$ be epi. Then f is split epi by ii), and split epis are preserved by any functor, in particular by the functor $(-)^X$. So X is internally projective.