Talk about Campana points on curves

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1 Campana points on curves

These lecture notes mainly follow Chapter 2 of notes by Abramovich [1], which in turn is largely based upon the work of Campana [2].
Let k be a numberfield and denote by $\mathcal{O} := \mathcal{O}_{k,S}$ the ring of integers away from S for some finite set of primes S. Throughout the document, a variety over k will mean a smooth proper integral scheme of finite type over k. A curve will mean a variety of dimension 1.

Definition 1. A *Campana pair* is a pair (Y/Δ), where Y is a curve and Δ is a $\mathbb{Q}$-divisor on Y of the form $\sum_y (1 - \frac{1}{m_y}) \cdot y$ for some $m_y \in \mathbb{Z}_{>0}$ with $m_y = 1$ for almost all y.

An especially important case of this will be the following.

Definition 2. Let $f : X \to Y$ be a dominant morphism of varieties over k where $\dim(Y) = 1$. For $p \in Y$ a closed point let $m_y = \min_i m_i$, where $f^*(y) = \sum_{m_i \neq 0} m_i \cdot C_i$ as divisors on X. Define the divisor Δ_f on Y by setting $\Delta_f := \sum_y \delta_y \cdot y$ with $\delta_y = 1 - \frac{1}{m_y}$.

The sum is indeed finite, which follows from [4, Tag 0574] Lemma 37.26.7. If $\mathcal{Y}$ is a model of Y that is proper over $\mathcal{O}$ and $\Delta = \sum_y \delta_y \cdot y$ is a divisor on Y with $\delta_y = 1 - \frac{1}{m_y}$, we will also write Δ for the divisor $\sum_y \delta_y \cdot \mathbf{y}$ on $\mathcal{Y}$, where $\mathbf{y}$ is the Zariski closure of y in $\mathcal{Y}$.

Definition 3. A point $y \in Y(k)$ is called a *soft integral point* on $(\mathcal{Y}/\Delta)$ if for any nonzero prime $\mathfrak{p} \subset \mathcal{O}$ and any integral point $\mathbf{z} \in \Delta$ such that $\mathbf{y}_\mathfrak{p} = \mathbf{z}_\mathfrak{p}$ in $\mathcal{Y}_\mathfrak{p} := \mathcal{Y} \times_\mathcal{O} \operatorname{Spec}(\mathbb{F}_\mathfrak{p})$, we have $\operatorname{mult}_\mathfrak{p}(\mathbf{y} \cap \mathbf{z}) \geq m_z$. Denote the set of soft integral points by $(\mathcal{Y}/\Delta)(\mathcal{O})$.

When $\Delta = 0$, we simply recover the rational points of Y. When $\Delta \geq \Delta'$ we have the inclusion $(\mathcal{Y}/\Delta)(\mathcal{O}) \subset (\mathcal{Y}/\Delta')(\mathcal{O})$. The multiplicity can be calculated as follows in the case that $(y)_\mathfrak{p} = (z)_\mathfrak{p}$.

Remark 4. Let $\mathcal{Y}$ be such a model and assume that $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}} := \mathcal{Y} \times_{\mathcal{O}} \operatorname{Spec}(\mathcal{O}_\mathfrak{p})$ is a regular scheme. The multiplicity $\operatorname{mult}_\mathfrak{p}(\mathbf{y} \cap \mathbf{z})$ equals by definition the integer $r_\mathfrak{p}$ such that $\mathbf{y} \cap \mathbf{z} = \operatorname{Spec}(\mathcal{O}/I)$ where $I = \mathfrak{p}^{r_\mathfrak{p}} \cdot J$ where J and $\mathfrak{p}$ are coprime ideals. Write $R = \mathcal{O}/I$, then note that $r_\mathfrak{p} = \operatorname{length}(R \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{p})$. By an abuse of notation we will write $\mathbf{z}$ and $\mathbf{y}$ for the $\mathcal{O}_\mathfrak{p}$ points on $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}}$. Since $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}}$ is regular, we may find an affine open $\operatorname{Spec}(A) = U \subset \mathcal{Y}$ containing $\mathbf{z}_\mathfrak{p}$ (and hence containing $\mathbf{z}$) such that the divisor $\mathbf{z}$ is cut out by a single regular function $g \in A$. Note that since $\mathbf{z}_\mathfrak{p} = \mathbf{y}_\mathfrak{p}$ we have that $\mathbf{y} \in U$. Then $\operatorname{length}(R \otimes_{\mathcal{O}} \mathcal{O}_\mathfrak{p}) = \operatorname{length}(\mathcal{O}_\mathfrak{p}/I_\mathfrak{p})$ equals the length of the global sections of the following fibre product:

$$\begin{array}{ccc} T & \longleftarrow & \mathcal{O}_\mathfrak{p} \\ \uparrow & & \uparrow{\scriptstyle \mathbf{z}^*} \\ \mathcal{O}_\mathfrak{p} & \underset{\mathbf{y}^*}{\longleftarrow} & A \end{array}$$

The map $\mathbf{z}^*$ being reduction modulo $g \in A$, we obtain that $T = \mathcal{O}_\mathfrak{p}/(\mathbf{y}^*(g)) = \mathcal{O}_\mathfrak{p}/(g(\mathbf{y}))$. Hence we see that $\operatorname{mult}_\mathfrak{p}(\mathbf{y} \cap \mathbf{z}) = \nu_\mathfrak{p}(f(\mathbf{y}))$ for f a local equation of $\mathbf{z}$ in $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}}$.

It is a classical result that there always exists a regular model $\mathcal{Y}$ of Y over $\mathcal{O}$ and in general a model $\mathcal{Y}$ becomes regular when base-changed to $\mathcal{O}_S$ for S large enough. Now we can 'compute' the Campana points in an example.

Example 5. Consider $Y = \mathbb{P}^1_k$ and $\Delta = \frac{4}{5} \cdot [0] + \frac{2}{3} \cdot [1] + \frac{2}{3} \cdot [\infty]$. Then $\frac{X_1}{X_0}$, $\frac{X_1 - X_0}{X_0}$ and $\frac{X_0}{X_1}$ are rational functions on $\mathbb{P}^1_{\mathcal{O}}$ that cut out $[0], [1], [\infty]$ locally in $\mathbb{P}^1_{\mathcal{O}_\mathfrak{p}}$ for all $\mathfrak{p}$. By the above remark $(\mathbb{P}_{\mathcal{O}}/\Delta)(\mathcal{O})$ then consists of the set of $[x : y] \in \mathbb{P}^1(\mathcal{O})$, where $x, y \in \mathcal{O}$ and for all $\mathfrak{p}$ of $\mathcal{O}$ we have:

$$\begin{cases} \nu_\mathfrak{p}(\frac{x}{y}) > 0 \implies \nu_\mathfrak{p}(\frac{x}{y}) \geq 5 \\ \nu_\mathfrak{p}(\frac{x-y}{y}) > 0 \implies \nu_\mathfrak{p}(\frac{x-y}{y}) \geq 3 \\ \nu_p(\frac{y}{x}) > 0 \implies \nu_\mathfrak{p}(\frac{y}{x}) \geq 3 \end{cases}$$

Remark 6. Although in the above example we were able to find **one** local equation for each of the relevant divisors, this need not always be the case. Even in the case $\mathcal{Y} = \mathbb{P}^1_{\mathcal{O}}$ with $\mathcal{O} = \mathcal{O}_k$ and $k = \mathbb{Q}(\sqrt{-5})$, the $\mathcal{O}$-point $[1 : \frac{\sqrt{-5}-1}{\sqrt{-5}+1}]$ does not admit $f \in k(\frac{x_0}{x_1})^\times$ that cuts it out locally in $\mathbb{P}^1_{\mathcal{O}_\mathfrak{p}}$ for all $\mathfrak{p}$.

The following proposition links the previous two definitions.

Proposition 7. *Let $f : X \to Y$ be a dominant morphism of smooth proper varieties over k with $\dim(Y) = 1$ and assume that there are regular models $\mathcal{X}$ and $\mathcal{Y}$ and a morphism $\tilde{f} : \mathcal{X} \to \mathcal{Y}$ extending f. Then for any $x \in X(k)$, $f(x)$ is a soft integral point on $(\mathcal{Y}, \Delta_f)$.*

Proof. Let $x \in X(k)$ such that $\tilde{f}(\mathbf{x}) =: \mathbf{y}$ has that $\mathbf{y}_\mathfrak{p} = \mathbf{z}_\mathfrak{p}$ for some $\mathbf{z} \in \Delta_f$ and some nonzero $\mathfrak{p} \subset \mathcal{O}$. Since $\mathcal{Y}$ is regular, so is $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}}$, so by the above remark we have that $\operatorname{mult}_\mathfrak{p}(\mathbf{y} \cap \mathbf{z}) = \nu_\mathfrak{p}(\mathbf{y}^*(g))$ for $g \in K(\mathcal{Y})^\times$ a local equation for $\mathbf{z}$ in $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}}$. Since $\mathbf{y}^* = \mathbf{x}^* \circ \tilde{f}^*$, this equals $\nu_\mathfrak{p}(\mathbf{x}^*(\tilde{f}^*(g)))$. The element $\tilde{f}^*(g) \in K(X)^\times$ satisfies $\nu_{C_i}(\tilde{f}^*(g)) = m_i$ for C_i a prime divisor on X. This implies that for $\mathcal{C}_i$ the Zariski closure of C_i in $\mathcal{Y}_{\mathcal{O}_\mathfrak{p}}$ we have $\nu_{\mathcal{C}_i}(\tilde{f}^*(g)) = m_i$ (since $\mathcal{O}_{\mathcal{X},\mathcal{C}_i} = \mathcal{O}_{X,C_i}$ and the valuations that they induce on $K(X)^\times$ are the same). This implies that the divisor $\tilde{f}^*([\mathbf{z}])$ is of the form $\sum_i m_i \cdot \mathcal{C}_i + m_{\mathfrak{p}} \cdot \mathcal{X}_{\mathbb{F}_{\mathfrak{p}}}$. If $m_{\mathfrak{p}} > 0$, then $\tilde{f}^*(\mathcal{X}_{\mathbb{F}_{\mathfrak{p}}}) = \mathbf{z}_{\mathfrak{p}}$ and hence $\tilde{f}$ is not surjective. But $\tilde{f}$ is dominant and proper, hence closed, so this can not be the case. Since $\tilde{f}^*([\mathbf{z}]) = \sum_i m_i \cdot \mathcal{C}_i$, we have that locally on U around $\mathbf{x}_{\mathfrak{p}}$, $\tilde{f}^*([\mathbf{z}])$ is given by $\operatorname{div}(\prod_i \mathbf{t}_i^{m_i})$ for $\mathbf{t}_i$ a local coordinate for $\mathcal{C}_i$ at $\mathbf{x}_{\mathfrak{p}}$ and hence up to an invertible function on U we have $\tilde{f}^*(g) = \prod_i \mathbf{t}_i^{m_i}$. Now since by assumption $\tilde{f}^*(g)(\mathbf{x}_{\mathfrak{p}}) = g(\mathbf{z}_{\mathfrak{p}}) = 0$, we obtain that $\tilde{f}^*(g)(\mathbf{x})$ vanishes modulo $\mathfrak{p}$. But then since $\tilde{f}^*(g) = \prod_i \mathbf{t}_i^{m_i}$ it vanishes with multiplicity at least m_z at $\mathbf{x}$. By Remark 4, we conclude that $\operatorname{mult}_{\mathfrak{p}}(\mathbf{y} \cap \mathbf{z}) \geq m_z$ and hence we conclude that $f(x)$ is a soft integral point on $(\mathcal{Y}/\Delta)(\mathcal{O})$. □

Such an extended morphism exists over $\mathcal{O}_S$ for S large enough.

2 Orbifold Mordell conjecture

We begin by defining the Kodaira dimension of a Campana pair (Y/Δ).

Definition 8. The *Canonical divisor* of (Y/Δ) is $K_{Y/\Delta} = K_Y + \Delta$. The *Kodaira dimension* of a Campana pair (Y/Δ) is the Itaka dimension of the $\mathbb{Q}$ divisor $K_{Y/\Delta}$. It is denoted $\kappa(Y/\Delta)$. We say that (Y/Δ) if of general type if $K_Y + \Delta$ is big.

For a curve this is relatively easily understood.

Remark 9. Let Y be a curve, then $K_Y + \Delta$ is big if and only if it is ample (because a rational map from a smooth curve to a projective variety extends uniquely). On a curve, a divisor D is very ample if and only if $h^0(Y, D - [x] - [y]) = h^0(Y, D) - 2$ for any closed points y, x, so by Riemann-Roch, D is ample if and only if its degree is positive. So we have the following cases:

$$\begin{cases} \kappa(Y/\Delta) = -\infty \text{ when } \deg(\Delta + K_Y) < 0 \\ \kappa(Y/\Delta) = 0 \text{ when } \deg(\Delta + K_Y) = 0 \\ \kappa(Y/\Delta) = 1 \text{ when } \deg(\Delta + K_Y) > 0 \end{cases}$$

We can now define morphisms between Campana pair.

Definition 10. A morphism of curves $g : Y \to Y'$ is called a morphism of Campana pairs $g : (Y/\Delta) \to (Y'/\Delta')$ if $K_{Y/\Delta} \geq g^*(K_{Y'/\Delta'})$ (for a suitable choice of canonical divisors on Y, Y').

Remark 11. This may seem like an odd definition. However if we look more closely we note that this is equivalent to asking that for all $y' \in Y'$ and any $y \in g^{-1}(y)$ we have that $n_{y'} \leq m_y \cdot n_y$, where m_y is the ramification at y, $\Delta = \sum_y (1 - \frac{1}{m_y}) \cdot y$ and $\Delta' = \sum_{y'} (1 - \frac{1}{m_{y'}}) \cdot y'$.

We have already encountered certain special cases of these.

Example 12. Let $g : Y \to Y'$ be a morphism of curves, then for $\Delta = 0$ and $\Delta' = \Delta_g$, g is a morphism $(Y/\Delta) \to (Y'/\Delta_g)$. Indeed, the 'canonical divisor version' of the Riemann Hurwicz theorem gives that $g^*(K'_C) + R$ is a canonical divisor on C for $R := \sum_{p \in C'} \sum_{q \in g^{-1}(p)} (m_q - 1) \cdot q$ the ramification divisor. It follows immediately that $R \geq g^*(\Delta_g)$ and hence we get the result.

We have the following proposition.

Proposition 13. *Let $f : (Y/\Delta) \to (C'/\Delta')$ be a morphism of Campana pairs and assume that f extends to a morphism $\tilde{f} : \mathcal{Y} \to \mathcal{Y}'$, where $\mathcal{Y}, \mathcal{Y}'$ are regular $\mathcal{O}$-models. Then g maps soft integral points as follows: $g((\mathcal{Y}/\Delta)(\mathcal{O})) \subset (\mathcal{Y}'/\Delta')(\mathcal{O})$.*

Proof. The proof is more or less the same as the one given for Proposition 7. One picks a point $\mathbf{y} \in \mathcal{Y}_{\mathcal{O}_\mathfrak{p}}(\mathcal{O}_\mathfrak{p})$ such that for all $\mathbf{z} \in \Delta$ with $\mathbf{z}_\mathfrak{p} = \mathbf{y}_\mathfrak{p}$, a local equation g for $\mathbf{z}$ has $\nu_\mathfrak{p}(g(\mathbf{x})) \geq m_z$. Assume that $\mathbf{y}' := g(\mathbf{y})$ has $\mathbf{y}'_\mathfrak{p} = \mathbf{z}'_\mathfrak{p}$ for some $\mathbf{z}' \subset \Delta'$. The proof of Proposition 7 gives us that for g' a local equation for $\mathbf{z}'$ that $\tilde{f}^*(g') = \prod_i t_i^{m_i}$ for t_i local equations for points in the fiber of $\mathbf{z}'$. Then $g'(\mathbf{y}') = \prod_i t_i^{m_i}(\mathbf{y})$ and since at least one of the t_i vanishes at $\mathbf{y}$, it does so with multiplicity $n_i = n_{p_i}$ for p_i the point at which t_i is a uniformizer. This yields $\nu_\mathfrak{p}(g'(\mathbf{y}')) \geq m_i \cdot n_{p_i} \geq m_z$ by Remark 11. □

The main point of this part is to state the following conjecture.

Conjecture 14 (Orbifold Mordell conjecture)**.** Let (Y/Δ) be a Campana pair of general type. Then $(\mathcal{Y}/\Delta)(\mathcal{O})$ is finite.

Note that we are still using the notation $\mathcal{O} = \mathcal{O}_{k,S}$ above for S arbitrary. The Campana mordell conjecture is true when Y itself is of general type by Faltings theorem.

Remark 15. The finiteness above does not depend on the choice of model. If $\mathcal{Y}$ is an $\mathcal{O}_S$ model such that $(\mathcal{Y}_{\mathcal{O}_{S'}}/\Delta)(\mathcal{O}_{S'})$ is finite for all S' containing S and $\mathcal{Y}'$ is another $\mathcal{O}$-model, then $\mathcal{Y}'$ and $\mathcal{Y}$ are isomorphic over $\mathcal{O}_{S'}$ for S' large enough from which follows that $(\mathcal{Y}_{\mathcal{O}_{S'}}/\Delta)(\mathcal{O}_{S'}) = (\mathcal{Y}'_{\mathcal{O}_{S'}}/\Delta)(\mathcal{O}_{S'})$.

The following Corollary related to the Bombieri-Lang conjecture follows from Orbifold Mordell.

Corollary 16. *Assume that Conjecture 14 is true. Let X be a smooth projective variety over a numberfield with a dominant morphism $f : X \to \mathbb{P}^1_k$ with $\deg(\Delta_f) > 2$. Then $X(k)$ does not lie Zariski dense in X.*

Proof. We have the equality $X(k) = \bigsqcup_{y \in f(X(k))} X_y(k)$, where X_y is the fibre over y. Pick a finite set S such that X admits a regular model over $\mathcal{O} := \mathcal{O}_{k,S}$ and such that f is defined over $\mathcal{O}$. The hypothesis $\deg(\Delta_f) > 2$ implies that $(\mathbb{P}^1_\mathcal{O}/\Delta)(\mathcal{O})$ is finite by Conjecture 14. By Proposition 7 we obtain that $f(X(k)) \subset (\mathbb{P}^1_\mathcal{O}/\Delta)(\mathcal{O})$ and thus $f(X(k))$ is finite. Hence $X(k) = \bigsqcup_{y \in f(X(k))} X_y(k)$ where the disjoint union is finite and hence $X(k)$ does not lie dense in X. □

We make the following definition for convenience.

Definition 17. For Y a curve, a divisor $\Delta = \sum_y (1 - \frac{1}{m_y}) \cdot y$ on Y is said to be of form $(m_1, ..., m_n)$ if $\mathrm{Supp}(\Delta)$ consists of n points y_i, with $m_{y_i} = m_i$.

One can do certain reductions to show that Conjecture 14 holds if it holds in only some cases.

Lemma 18. *To prove Conjecture 14 it suffices to do so for $Y = \mathbb{P}^1_k$ and Δ of one of the following types:*

$$(2,2,2,2,2) \text{ or } (2,2,2,3) \text{ or } (3,3,4) \text{ or } (2,4,5) \text{ or } (2,3,7)$$

Proof. As remarked, we do not have to consider the case $g(Y) \geq 2$. If $g(Y) = 1$, we may assume that $\Delta = \frac{1}{2} \cdot p$. then Y has a 2 to 1 cover $f : Y \to \mathbb{P}^1$, ramified over 4 points p_1, p_2, p_3, p_4 and we may assume that $p_1 = p$. Denote by q_i their images in $\mathbb{P}^1$. Then note that for $\Delta' = \frac{3}{4} p_1 + \sum_{i=2}^{4} \frac{1}{2} p_i$, f is a morphism $(Y/\Delta) \to (\mathbb{P}^1/\Delta')$. The morphism is defined over $\mathcal{O}_{k,S}$ for S large enough and hence by Proposition 13, the case $g(Y) = 1$ reduces to the case $Y = \mathbb{P}^1$ and $\Delta = (2,2,2,4)$.
In the case that $Y = \mathbb{P}^1$ it is a simple computation to reduce to these cases, using the fact that $(\mathbb{P}^1_{\mathcal{O}}/\Delta)(\mathcal{O}) \subset (\mathbb{P}^1_{\mathcal{O}}, \Delta')$ whenever $\Delta \geq \Delta'$. □

We end with the statement that the Orbifold-Mordell conjecture is implied by the abc-conjecture.

Conjecture 19 (*abc*-conjecture)**.** Consider triples $(a,b,c) \in \mathbb{Z}^3$ such that $c = a+b$ and $\gcd(a,b,c) = 1$. For all $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that any triple as above satisfies:

$$\max\{|a|, |b|, |c|\} < C_\epsilon \cdot \mathrm{Rad}(abc)^{1+\epsilon}$$

Indeed we have the following Theorem.

Proposition 20. *Assume that Conjecture 19 is true. Then so is Conjecture 14.*

The general case follows from the methods developed by Elkies [3] as claimed in [1] p.37. We give the proof in the case that $k = \mathbb{Q}$ and the support of Δ is 3 points over k.

In this case we may assume that the points are $0, 1, \infty$ and defined my multiplicities m_0, m_1, m_∞ at $0, 1, \infty$ respectively (otherwise use a transformation in $\mathrm{PGL}_2(\mathbb{Q})$). We have to show that there exist only finitely many $[a : c]$ with $a, c \in \mathbb{Z}$ coprime such that for $b := a - c$ the following hold:

$$\begin{cases} p|a \implies p^{m_0}|a \\ p|b \implies p^{m_1}|b \\ p|c \implies p^{m_\infty}|c \end{cases} \tag{1}$$

For $x \in \{a, b, c\}$ this implies that $|x|^{1/m_i} \geq \mathrm{Rad}(x)$ for the $i \in \{0, 1, \infty\}$ corresponding to a, b or c. Set $M := \max\{|a|, |b|, |c|\}$, then we get $M^{1/m_0 + 1/m_1 + 1/m_\infty} \geq \mathrm{Rad}(abc)$. That $(\mathbb{P}^1/\Delta)$ is of general type is equivalent to $1/m_0+1/m_1+1/m_\infty < 1$. Pick any $0 < \epsilon < 1-(1/m_0+1/m_1+1/m_\infty)$. Since $0 < \epsilon < 1$, it follows from Conjecture 19 that there is a constant $C > 0$ such that all coprime (a, b, c) with $c = a + b$ satisfy $M^{1-\epsilon} < C \cdot \mathrm{Rad}(abc)$. Hence for our a, b, c satisfying (1) this gives that $M^{1-(1/m_0+1/m_1+1/m_\infty+\epsilon)} < C$. In particular the absolute value of a and c is bounded giving that there are only finitely many options.

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