Generalized Cox rings over arbitrary fields

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Origin

The close relation between Cox rings and universal torsors for toric varieties is well known since the work of Colliot-Thélène and Sansuc (1970s) and Cox (1990s).

$$\mathbb{A}_k^{n+1} \smallsetminus \{0\} \subseteq \operatorname{Spec} k[x_0, \dots, x_n]$$
$$\downarrow$$
$$\mathbb{P}_k^n$$

More recently, it has been established for locally factorial varieties over algebraically closed fields making use of the additional notion of Cox sheaf [1].

Universal torsors are defined over arbitary fields. What about Cox rings? Universal torsors are special torsors under quasitori. Are there Cox rings related to arbitrary torsors under quasitori?

Goal

To define Cox rings and Cox sheaves related to torsors under quasitori over arbitrary fields, to study their existence and classification, and their arithmetic applications.

Setting

Let k be a field, $\mathfrak{g} = \operatorname{Gal}(\bar{k}/k)$. There is an equivalence of categories

$$\{\text{Quasitori over } k\} \longrightarrow \{\text{Finitely generated } \mathfrak{g}\text{-modules}^*\} =: \mathcal{M}$$
$$G \longmapsto \widehat{G} := \operatorname{Hom}_{\bar{k}}(G_{\bar{k}}, \mathbb{G}_{m,\bar{k}})$$
$$\operatorname{Spec} \bar{k}[M]^{\mathfrak{g}} =: \widehat{M} \longleftarrow M.$$

Let X be a geometrically integral k-variety, $\bar{k}[X]^\times = \bar{k}^\times$. By [2], torsors over X under a quasitorus G are classified by

$$0 \to H^1_{ét}(k, G) \to H^1_{ét}(X, G_X) \overset{type}{\to} \operatorname{Hom}_{\mathfrak{g}}(\widehat{G}, \operatorname{Pic}(X_{\bar{k}})).$$

Let $\lambda : M \to \operatorname{Pic}(X_{\bar{k}})$ be a morphism of $\mathfrak{g}$-modules with $M \in \mathcal{M}$.

Definitions

over $\bar{k}$

- A *Cox sheaf* of $X_{\bar{k}}$ *of type* λ is an M-graded $\mathcal{O}_{X_{\bar{k}}}$-algebra structure on $\bigoplus_{m \in M} \mathcal{O}_{X_{\bar{k}}}(D_m)$, where $[D_m] = \lambda(m)\ \forall\ m \in M$.
- A *Cox ring* of $X_{\bar{k}}$ *of type* λ is an M-graded $\bar{k}$-algebra structure on $\bigoplus_{m \in M} H^0(X_{\bar{k}}, \mathcal{O}_{X_{\bar{k}}}(D_m))$ compatible with sum of divisors.

over k

- A *Cox sheaf* of X *of type* λ is an $\mathcal{O}_X$-algebra $\mathcal{R}$ such that $\mathcal{R} \otimes_k \bar{k}$ is a Cox sheaf of $X_{\bar{k}}$ of type λ with compatible $\mathfrak{g}$-action.
- A *Cox ring* of X *of type* λ is a k-algebra R such that $R \otimes_k \bar{k}$ is a Cox ring of $X_{\bar{k}}$ of type λ with compatible $\mathfrak{g}$-action.

*without p-torsion if $\operatorname{char}(k) = p > 0$

Construction 1

Fix generators $m_1, \dots, m_n$ of M, and divisors $D_1, \dots, D_n$ such that $[D_i] = \lambda(m_i)$. Let $\Lambda := \mathbb{Z}D_1 \oplus \dots \oplus \mathbb{Z}D_n$, and Λ_0 the kernel of $\varphi : \Lambda \twoheadrightarrow M$. Let $\chi : \Lambda_0 \to \bar{k}(X_{\bar{k}})^\times$ such that $\operatorname{div}(\chi(E)) = E$, $\forall\ E \in \Lambda_0$. Then

$$\left(\bigoplus_{D \in \Lambda} \mathcal{O}_{X_{\bar{k}}}(D)\right) \Big/ (1 - \chi(E) : E \in \Lambda_0)$$

is a Cox sheaf of $X_{\bar{k}}$ of type λ.

Uniqueness over $\bar{k}$

Up to isomorphism, there is exactly one Cox sheaf and exactly one Cox ring of $X_{\bar{k}}$ of type λ.

Automorphisms

- The group of automorphisms of a Cox sheaf of $X_{\bar{k}}$ of type λ is isomorphic to $\widehat{M}(\bar{k})$.
- The group of automorphisms of a Cox ring of $X_{\bar{k}}$ of type λ is isomorphic to $\widehat{M_{\text{eff}}}(\bar{k})$.

Pull-back

If $\varphi : M' \to M$ is a morphism of $\mathfrak{g}$-modules with $M' \in \mathcal{M}$, and $\mathcal{R} = \bigoplus_{m \in M} \mathcal{R}_m$ is a Cox sheaf of $X_{\bar{k}}$ of type λ, then $\varphi^*\mathcal{R} := \bigoplus_{m' \in M'} \mathcal{R}_{\varphi(m')}$ is a Cox sheaf of $X_{\bar{k}}$ of type $\lambda \circ \varphi$.

Construction 2

Assume that the exact sequence

$$1 \to \bar{k}^\times \to \bar{k}(X)^\times \to \bar{k}(X)^\times/\bar{k}^\times \to 1 \tag{1}$$

has a Galois equivariant splitting $\sigma : \bar{k}(X)^\times \to \bar{k}^\times$. For each $m \in M$, let $D_m \in \operatorname{CaDiv}(X_{\bar{k}})$ such that $[D_m] = \lambda(m)$. Define multiplication of homogeneous sections s_i of degree $m_i \in M$ by $s_1 \cdot s_2 = s_1 s_2 f$, where $f \in \bar{k}(X)^\times$ is the unique element such that $D_{m_1} + D_{m_2} + \operatorname{div}(f) = D_{m_1+m_2}$ and $\sigma(f) = 1$. Then the sheaf

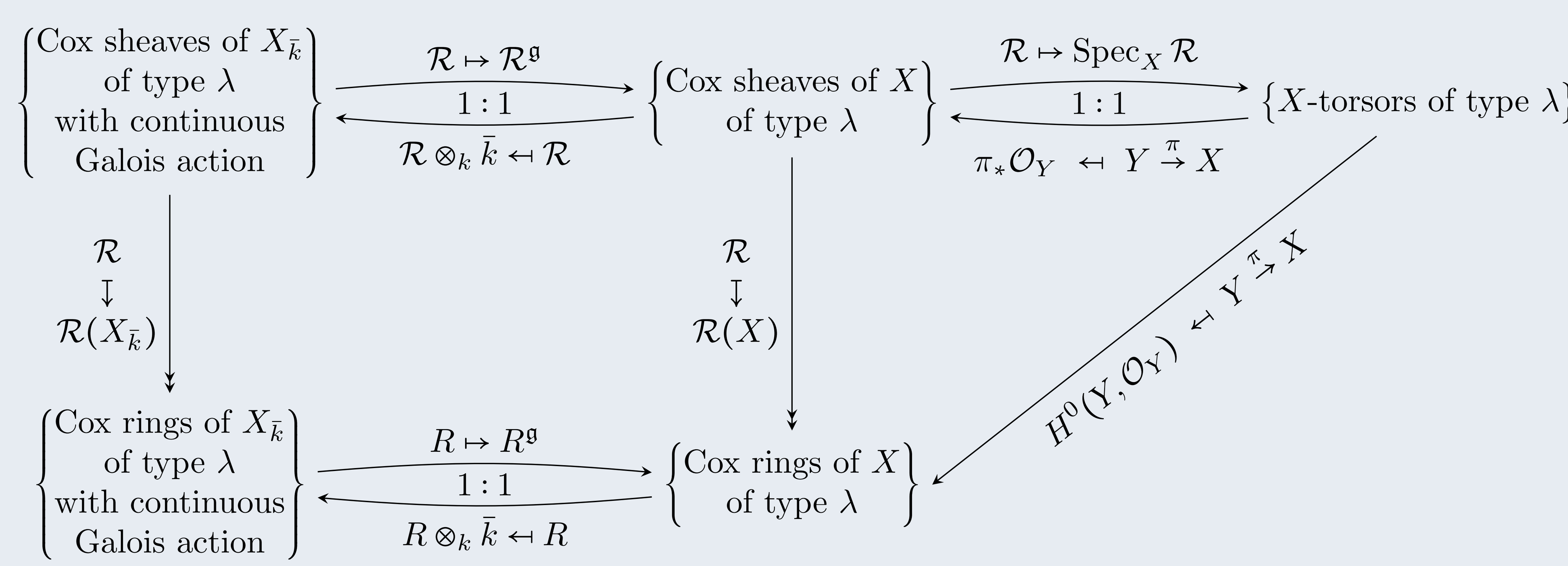
$$\bigoplus_{m \in M} \mathcal{O}_{X_{\bar{k}}}(D_m)$$

is a Cox sheaf of $X_{\bar{k}}$ of type λ with a continuous Galois action.

Existence over k

- [2] For X smooth, universal torsors of X exist if and only if (1) has a Galois equivariant splitting.
- Cox sheaves and Cox rings of X of arbitrary type exist if
  - (1) has a Galois equivariant splitting, or
  - $X(k) \neq \varnothing$.
- Cox sheaves and Cox rings of X of type λ exist if
  - the $\mathfrak{g}$-action on M is trivial, or
  - there exists a Cox sheaf of X of type $\lambda(M) \subseteq \operatorname{Pic}(X_{\bar{k}})$.
- Cox sheaves and Cox rings of X of type λ exist if and only if there exists a $\mathfrak{g}$-equivariant χ in Construction 1.

Classification over k



- Isomorphism classes of Cox sheaves of X of type λ are classified by $H^1_{ét}(k, \widehat{M})$.
- If $\lambda(M)$ is generated by effective divisors, then the vertical functors are essentially injective.

Quasi affine torsors

If $\lambda(M)$ contains an ample divisor class and X has a Cox sheaf $\mathcal{R}$ of type λ such that $\mathcal{R}(X)$ is a finitely generated k-algebra, then the natural morphism

$$\operatorname{Spec}_X \mathcal{R} \to \operatorname{Spec} \mathcal{R}(X) \tag{2}$$

is an $\widehat{M}$-equivariant open immersion.

Parameterization property

[2] If $\pi : Y \to X$ is a torsor of type λ, then

$$X(k) = \bigsqcup_{[\alpha] \in H^1(k, \widehat{M})} \pi^\alpha(Y^\alpha(k)), \tag{3}$$

where $\pi^\alpha : Y^\alpha \to X$ is a twist of Y.

Identity type: $\lambda = \operatorname{id}_{\operatorname{Pic}(X_{\bar{k}})}$

- Torsors of identity type are universal torsors, which have been introduced and studied by Colliot-Thélène and Sansuc [2].
- Construction 1 has been performed also using Weil divisors [1] instead of Cartier divisors to produce $\operatorname{Cl}(X_{\bar{k}})$-graded Cox rings.
- Smooth projective varieties over $\bar{k}$ with finitely generated Cox ring are called Mori dream spaces [3] and play an important role in the minimal model program.

Arithmetic applications

- Because of (3), universal torsors are used to investigate the existence of rational points on varieties over number fields (e.g. [2]).
- Because of (2) they have been used also to study the distribution of rational points on varieties with trivial $\mathfrak{g}$-action on $\operatorname{Pic}(X_{\bar{k}})$ (e.g. [4]). We observed that for varieties with nontrivial $\mathfrak{g}$-action on $\operatorname{Pic}(X_{\bar{k}})$, a similar role is played by torsors of type $\operatorname{Pic}(X) \hookrightarrow \operatorname{Pic}(X_{\bar{k}})$.

References

[1] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface. *Cox rings*, volume 144 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.

[2] J.-L. Colliot-Thélène and J.-J. Sansuc. La descente sur les variétés rationnelles. II. *Duke Math. J.*, 54(2):375–492, 1987.

[3] Y. Hu and S. Keel. Mori dream spaces and GIT. *Michigan Math. J.*, 48:331–348, 2000. Dedicated to William Fulton on the occasion of his 60th birthday.

[4] P. Salberger. Tamagawa measures on universal torsors and points of bounded height on Fano varieties. *Astérisque*, (251):91–258, 1998. Nombre et répartition de points de hauteur bornée (Paris, 1996).