# Integral models of generalised del Pezzo surfaces and their universal torsors


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# Setting

**Definition**

A generalised del Pezzo surface is a smooth projective rational surface with anticanonical divisor big and nef.

Let $S$ be a (possibly singular) del Pezzo surface whose minimal desingularization $X$ is a generalised del Pezzo surface.

We assume that $S$ is defined over a number field $K$ and $X$ is split, i.e. with trivial Galois action on its Picard group.

# Manin's conjecture

Assume that $-K_S$ is very ample.
Let $\phi : S \hookrightarrow \mathbb{P}^n_K$ be an anticanonical embedding and

$$H : S(K) \to \mathbb{R}, \quad x \mapsto \phi(x) = (x_0 : \ldots : x_n) \mapsto \prod_{\nu \in \Omega_K} \sup_{i=0,\ldots,n} |x_i|_\nu$$

the induced anticanonical height function.

## Manin's conjecture for $S$

There is an open subset $U$ of $S$ and a positive constant $c$ such that

$$\#\{x \in U(K) : H(x) \le B\} \sim cB(\log B)^{r-1}, \quad B \to \infty,$$

where $r$ is the Picard number of $X$.

# Universal torsor method

- Parameterize $U(K)$ by integral points on a universal torsor $Y$ of $X$.
- Lift the height function to $Y$ and count lattice points of bounded height.

# Cox rings and universal torsors

The Cox ring of $X$ is the $\mathrm{Pic}(X)$-graded ring

$$\mathrm{Cox}(X) = \bigoplus_{[D]\in \mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D)) = K[\eta_1, \dots, \eta_N]/(g_1, \dots, g_s).$$

The universal torsor of $X$ is an open subset $Y$ of $\mathrm{Spec}(\mathrm{Cox}(X))$, whose complement is defined by monic monomial equations $f_1, \dots, f_m$ of the same degree, endowed with a morphism

$$\pi : Y \to X$$

which is a geometric quotient for the action of $\mathbb{G}_K^r$ induced by the $\mathrm{Pic}(X)$-grading on $\mathrm{Cox}(X)$.

# Parameterization (aim)

Assume that there is a commutative diagram

$$\begin{array}{ccccccccccc}
\mathbb{A}^N_K & \supset & Y & \xrightarrow{\pi} & X & \xrightarrow{\psi} & S & \supset & U & & K \\
\downarrow & & \downarrow & & \downarrow & & & & & & | \\
\mathbb{A}^N_{\mathcal{O}_K} & \supset & \widetilde{Y} & \xrightarrow{\tilde{\pi}} & \widetilde{X} & & & & & & \mathcal{O}_K
\end{array}$$

with $\psi^{-1}(U) \cong U$.

- If $\widetilde{X}$ is proper, $X(K) = \widetilde{X}(\mathcal{O}_K)$.
- If $\tilde{\pi}$ is a torsor, $\widetilde{X}(\mathcal{O}_K) = \bigsqcup_{\mathfrak{c}\in\mathcal{C}^r} {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K)/\mathbb{G}^r_{\mathcal{O}_K}(\mathcal{O}_K)$, for a system $\mathcal{C}$ of representatives for the ideal classes of $\mathcal{O}_K$.

Then $U(K) = \bigsqcup_{\mathfrak{c}\in\mathcal{C}^r} \left( {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K) \cap \pi^{-1}(\psi^{-1}(U(K))) \right) /\mathbb{G}^r_{\mathcal{O}_K}(\mathcal{O}_K)$.

- If $\widetilde{X}$ is split, smooth, with geometrically integral fibers, we can use it to verify Peyre's prediction on the constant $c$.

# Basic example

Take $X = S = \mathbb{P}^2_K$ and $U = \{x_0 x_1 x_2 \neq 0\}$.
Then we have a commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{A}^3_K & \supset & \mathbb{A}^3_K \setminus \{0\} & \xrightarrow{\pi} & \mathbb{P}^2_K & \xrightarrow{\psi = \mathrm{id}} & \mathbb{P}^2_K & \supset & U & & K \\
\downarrow & & \downarrow & & \downarrow & & & & & & | \\
\mathbb{A}^3_{\mathcal{O}_K} & \supset & \mathbb{A}^3_{\mathcal{O}_K} \setminus \{0\} & \xrightarrow{\tilde{\pi}} & \mathbb{P}^2_{\mathcal{O}_K} & & & & & & \mathcal{O}_K
\end{array}$$

- $\widetilde{X} = \mathbb{P}^2_{\mathcal{O}_K}$ is proper over $\mathcal{O}_K$.
- $\tilde{\pi}$ is a universal torsor.
- $\widetilde{X} = \mathbb{P}^2_{\mathcal{O}_K}$ is split, smooth, with geometrically integral fibers over $\mathcal{O}_K$.

# Construction

Let $\mathcal{O}_K$ be the ring of integers of $K$.
$\mathrm{Cox}(X) = K[\eta_1, \ldots, \eta_N]/(g_1, \ldots, g_s)$.
We assume that $g_1, \ldots, g_s$ have integral coefficients
and that $(g_1, \ldots, g_s) \cap \mathcal{O}_K[\eta_1, \ldots, \eta_N] = (g_1, \ldots, g_s)$.
Let

- $R := \mathcal{O}_K[\eta_1, \ldots, \eta_N]/(g_1, \ldots, g_s)$,
- $\widetilde{Y} := \mathrm{Spec}(R) \smallsetminus V(f_1, \ldots, f_m) = \bigcup_{i=1}^m \mathrm{Spec}(R_{f_i})$,
- $\widetilde{X}$ the gluing of $\mathrm{Spec}(R_{f_i,0})$, $i = 1, \ldots, m$, and
- $\tilde{\pi} : \widetilde{Y} \to \widetilde{X}$, the morphism induced by the inclusions $R_{f_i,0} \subset R_{f_i}$.

Then the following diagram commutes

$$\begin{array}{ccccccc}
\mathbb{A}^N_K & \supset & Y & \xrightarrow{\pi} & X & & K \\
\downarrow & & \downarrow & & \downarrow & & | \\
\mathbb{A}^N_{\mathcal{O}_K} & \supset & \widetilde{Y} & \xrightarrow{\tilde{\pi}} & \widetilde{X} & & \mathcal{O}_K
\end{array}$$

# First properties

**Proposition (Being a torsor)**

If the degrees of the homogeneous invertible elements of $R_{f_i}$ generate $\mathrm{Pic}(X)$ for all $i = 1, \dots, m$, then $\tilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is a torsor under $\mathbb{G}^r_{\mathcal{O}_K}$.

**Proposition (Integrality)**

If $(g_1, \dots, g_s)(R \otimes_{\mathcal{O}_K} \overline{k(\mathfrak{p})})$ is prime for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, then $\widetilde{X}$ has geometrically integral fibers over $\mathcal{O}_K$.

**Corollary (Smoothness)**

If, moreover, the Jacobian matrix $\left(\frac{\partial g_i}{\partial \eta_j}(y)\right)_{\substack{1 \leq i \leq s \\ 1 \leq j \leq N}}$ has rank $N - \dim X - r$ for all $y \in \widetilde{Y}(\overline{k(\mathfrak{p})})$ and all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, then $\widetilde{X}$ is smooth over $\mathcal{O}_K$.

# Projectivity

**Proposition (Projectivity)**

Assume that the degree of $f_1, \ldots, f_m$ is ample. Let $C_K$ and $C_{\mathcal{O}_K}$ be the ideals of $K[\eta_1, \ldots, \eta_N]$ and $\mathcal{O}_K[\eta_1, \ldots, \eta_N]$ generated by $f_1, \ldots, f_m, g_1, \ldots, g_s$. If $\sqrt{C_K} \cap \mathcal{O}_K[\eta_1, \ldots, \eta_N] = \sqrt{C_{\mathcal{O}_K}}$, then $\widetilde{X}$ is projective over $\mathcal{O}_K$.

# Blowing-up

Assume that $\eta_N$ defines a $(-1)$-curve on $X$ and let $b : X \to X'$ be its contraction. Assume that the center of $b$ is the intersection of the images of the prime divisors corresponding to $\eta_1, \eta_2$, and that $(\eta_N - 1, g_1, \dots, g_s) \cap \mathcal{O}_K[\eta_1, \dots, \eta_N] = (\eta_N - 1, g_1, \dots, g_s)$.
Let $R' := \mathcal{O}_K[\eta_1, \dots, \eta_N]/(\eta_N - 1, g_1, \dots, g_s)$ and take monic monomials $f'_1, \dots, f'_{m'}$ defining the complement of the universal torsors of $X'$ in $\mathrm{Spec}(\mathrm{Cox}(X'))$, such that the degrees of the invertible homogeneous elements of $R'_{f'_i}$ generate $\mathrm{Pic}(X')$ for all $i = 1, \dots, m'$, and the ideal generated by $f_1, \dots, f_m$ has the same radical as the ideal generated by $b^* f'_1 \eta_1, \dots, b^* f'_{m'} \eta_2$, where $b^* : R' \to R, \eta_i \to \begin{cases} \eta_i \eta_N & \text{if } i = 1, 2; \\ \eta_i & \text{otherwise.} \end{cases}$

Then $\widetilde{X}$ is a blowing-up of the model $\widetilde{X}'$ defined by $f'_1, \dots, f'_{m'}$ with center the closed subscheme defined by $\eta_1, \eta_2$.

# A del Pezzo surface of degree 4

Let $S \subset \mathbb{P}^4_K$ be defined by $x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0$.

$S$ is an anticanonically embedded del Pezzo surface with singularities of type $A_3$ in $(0:0:0:0:1)$ and $A_1$ in $(0:1:0:0:0)$.

Let $U$ be the complement of the three lines in $S$:

$$x_0 = x_1 = x_2 = 0, \quad x_0 = x_2 = x_3 = 0, \quad x_1 = x_2 = x_3 = 0.$$

The minimal desingularization $\psi : X \to S$ restricts to an isomorphism $\psi^{-1}(U) \to U$.

Then it sufficies to parameterize $\psi^{-1}(U)$.

# Cox ring of $X$

$$\mathrm{Cox}(X) = K[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7).$$

For $1 \leq i \leq 9$, let $E_i$ be the prime divisor on $X$ defined by $\eta_i$.
Then

$$[E_1] = \ell_5, \quad [E_2] = \ell_4, \quad [E_3] = \ell_1 - \ell_2, \quad [E_4] = \ell_2 - \ell_3, \quad [E_5] = \ell_3,$$
$$[E_6] = \ell_0 - \ell_1 - \ell_4 - \ell_5, \quad [E_7] = \ell_0 - \ell_1 - \ell_2 - \ell_3,$$
$$[E_8] = \ell_0 - \ell_4, \quad [E_9] = \ell_0 - \ell_5,$$

for a chosen basis $\ell_0, \dots, \ell_5$ of $\mathrm{Pic}(X)$.

# How to compute $\mathrm{Spec}(\mathrm{Cox}(X)) \smallsetminus Y$

First computation.

**Proposition [ADHL10]**

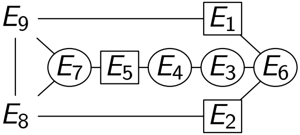
The ideal generated by $H^0(X, \mathcal{O}_X(D))$ defines $\mathrm{Spec}(\mathrm{Cox}(X)) \smallsetminus Y$ for any ample divisor $D$.

- Find an ample divisors $D$ on $X$,
  e.g. $[D] = 9\ell_0 - 3\ell_1 - 2\ell_2 - \ell_3 - \ell_4 - \ell_5$.
- Find all monomials of degree $[D]$ in $\eta_1, \ldots, \eta_9$.

Unfortunately, usually this choice of monomials do not assure that the model is a torsor.

# Dynkin diagram of $E_1, \dots, E_9$

For any $i \neq j$ the number of edges between $E_i$ and $E_j$ is the intersection number $[E_i].[E_j]$.



# How to compute $\mathrm{Spec}(\mathrm{Cox}(X)) \smallsetminus Y$

Second computation.

- Compute generators of $\prod_{\substack{1\leq i<j\leq 9 \\ \overline{E}_i.\overline{E}_j=0}} (\eta_i, \eta_j)$.
- For each of them consider the product of variables that appear in it (i.e. forget about the powers), and let $J$ be the ideal they generate.
- Choose a set of generators of $J$,
  e.g.

  $$A_{1,6}, \quad A_{1,9}, \quad A_{2,6}, \quad A_{2,8}, \quad A_{3,4}, \quad A_{3,6}, \quad A_{4,5}, \quad A_{5,7}, \quad A_{7,8,9}.$$

  where $A_{i_1,\dots,i_t} := \prod_{j\notin\{i_1,\dots,i_t\}} \eta_j$.
- Check that $J$ defines $\mathrm{Spec}(\mathrm{Cox}(X)) \smallsetminus Y$,
  e.g. $\sqrt{J} = \sqrt{(H^0(X, \mathcal{O}_X(D)))}$.

# $\tilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is a torsor

**Proposition (Being a torsor)**

If the degrees of the homogeneous invertible elements of $R_{f_i}$ generate $\mathrm{Pic}(X)$ for all $i = 1, \ldots, m$, then $\tilde{\pi} : \widetilde{Y} \to \widetilde{X}$ is a torsor under $\mathbb{G}^r_{\mathcal{O}_K}$.

For every $A_{i_1,\ldots,i_t}$ among

$$A_{1,6}, \quad A_{1,9}, \quad A_{2,6}, \quad A_{2,8}, \quad A_{3,4}, \quad A_{3,6}, \quad A_{4,5}, \quad A_{5,7}, \quad A_{7,8,9}.$$

check that

$$\ell_0, \ldots, \ell_5 \in \langle [E_j] : j \notin \{i_1, \ldots, i_t\} \rangle \subseteq \mathrm{Pic}(X).$$

Tool: *find integer solutions to 6 linear equations.*

# $\widetilde{X}$ has geometrically integral fibers over $\mathcal{O}_K$

**Proposition (integrality)**

If $(g_1, \ldots, g_s)(R \otimes_{\mathcal{O}_K} \overline{k(\mathfrak{p})})$ is prime for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, then $\widetilde{X}$ has geometrically integral fibers over $\mathcal{O}_K$.

Let $g := \eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7$.

- $g$ is irreducible over $\overline{k(\mathfrak{p})}$ for all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$.

# $\widetilde{X}$ is smooth

## Corollary (smoothness)

If, moreover, the Jacobian matrix $\left(\frac{\partial g_i}{\partial \eta_j}(y)\right)_{\substack{1\leq i\leq s\\ 1\leq j\leq N}}$ has rank $N - \dim X - r$ for all $y \in \widetilde{Y}(\overline{k(\mathfrak{p})})$ and all prime ideals $\mathfrak{p}$ of $\mathcal{O}_K$, then $\widetilde{X}$ is smooth over $\mathcal{O}_K$.

- Compute $\left(\frac{\partial g}{\partial \eta_j}(y)\right)_{1\leq j\leq N}$:

  $$(\eta_9,\ \eta_8,\ \eta_4^2\eta_5^3\eta_7,\ 2\eta_3\eta_4\eta_5^3\eta_7,\ 3\eta_3\eta_4^2\eta_5^2\eta_7,\ 0,\ \eta_3\eta_4^2\eta_5^3,\ \eta_2,\ \eta_1).$$

- Check that the rank could vanish only outside $Y$,
  i.e. $J \subseteq (\eta_1, \eta_2, \eta_8, \eta_9) \subseteq \sqrt{\left(\frac{\partial g}{\partial \eta_1}, \dots, \frac{\partial g}{\partial \eta_9}\right)}$.

# $\widetilde{X}$ is projective

## Proposition (Projectivity)

Assume that the degree of $f_1, \dots, f_m$ is ample. Let $C_K$ and $C_{\mathcal{O}_K}$ be the ideals of $K[\eta_1, \dots, \eta_N]$ and $\mathcal{O}_K[\eta_1, \dots, \eta_N]$ generated by $f_1, \dots, f_m, g_1, \dots, g_m$. If $\sqrt{C_K} \cap \mathcal{O}_K[\eta_1, \dots, \eta_N] = \sqrt{C_{\mathcal{O}_K}}$, then $\widetilde{X}$ is projective over $\mathcal{O}_K$.

Let $C'_K$ and $C'_{\mathcal{O}_K}$ be the ideals of $K[\eta_1, \dots, \eta_9]$ and $\mathcal{O}_K[\eta_1, \dots, \eta_9]$ generated by

$$A_{1,6}, \quad A_{1,9}, \quad A_{2,6}, \quad A_{2,8}, \quad A_{3,4}, \quad A_{3,6}, \quad A_{4,5}, \quad A_{5,7}, \quad A_{7,8,9}, \quad g.$$

- Check $C'_K \cap \mathcal{O}_K[\eta_1, \dots, \eta_9] = C'_{\mathcal{O}_K}$ by Gröbner basis computation.

# $\widetilde{X}$ is split

The contraction $b : X \to X'$ of $E_1$ yelds a toric variety $X'$ with Cox ring $K[\eta_1, \dots, \eta_9]/(\eta_1 - 1, \eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7) \cong K[\eta_2, \dots, \eta_8]$.

$$b^* : K[\eta_2, \dots, \eta_8] \to \mathrm{Cox}(X), \quad \eta_i \mapsto \begin{cases} \eta_i & \text{if } i \neq 6; \\ \eta_1\eta_6 & \text{if } i = 6. \end{cases}$$

Let $f_1', \dots, f_7'$ be the canonical monomials defining the universal torsor of $X'$.

- The degrees of the invertible homogeneous elements of $R'_{f_i'}$ generate $\mathrm{Pic}(X')$, by properties of toric varieties.
- $b^* f_1'\eta_6, \dots, b^* f_7'\eta_6, b^* f_1'\eta_9, \dots, b^* f_7'\eta_9 \in \sqrt{(H^0(X, \mathcal{O}_X(D)))}$.
- $f_1, \dots, f_9 \in \sqrt{(b^* f_1'\eta_6, \dots, b^* f_7'\eta_6, b^* f_1'\eta_9, \dots, b^* f_7'\eta_9)}$.

Then $\widetilde{X}$ is a blowing-up of $\widetilde{X}'$ with center the closed subscheme defined by $\eta_6, \eta_9$, and the same holds after base change to the residue fields $k(\mathfrak{p})$ of $\mathcal{O}_K$.

# Blowing-up

Assume that $\eta_N$ defines a $(-1)$-curve on $X$ and let $b : X \to X'$ be its contraction. Assume that the center of $b$ is the intersection of the images of the prime divisors corresponding to $\eta_1, \eta_2$, and that $(\eta_N - 1, g_1, \dots, g_s) \cap \mathcal{O}_K[\eta_1, \dots, \eta_N] = (\eta_N - 1, g_1, \dots, g_s)$.
Let $R' := \mathcal{O}_K[\eta_1, \dots, \eta_N]/(\eta_N - 1, g_1, \dots, g_s)$ and take monic monomials $f'_1, \dots, f'_{m'}$ defining the complement of the universal torsors of $X'$ in Spec(Cox($X'$)), such that the degrees of the invertible homogeneous elements of $R'_{f'_i}$ generate Pic($X'$) for all $i = 1, \dots, m'$, and the ideal generated by $f_1, \dots, f_m$ has the same radical as the ideal generated by $b^* f'_1 \eta_1, \dots, b^* f'_{m'} \eta_2$, where $b^* : R' \to R, \eta_i \mapsto \begin{cases} \eta_i \eta_N & \text{if } i = 1, 2; \\ \eta_i & \text{otherwise.} \end{cases}$

Then $\widetilde{X}$ is a blowing-up of the model $\widetilde{X}'$ defined by $f'_1, \dots, f'_{m'}$ with center the closed subscheme defined by $\eta_1, \eta_2$.

# Parameterization

$$
\begin{array}{ccccccccccc}
\mathbb{A}^N_K & \supset & Y & \xrightarrow{\pi} & X & \xrightarrow{\psi} & S & \supset & U & \qquad & K \\
\downarrow & & \downarrow & & \downarrow & & & & & & | \\
\mathbb{A}^N_{\mathcal{O}_K} & \supset & \widetilde{Y} & \xrightarrow{\tilde{\pi}} & \widetilde{X} & & & & & & \mathcal{O}_K
\end{array}
$$

and $\psi^{-1}(U) \cong U$.

- Since $\widetilde{X}$ is proper, $X(K) = \widetilde{X}(\mathcal{O}_K)$.
- Since $\tilde{\pi}$ is a torsor, $\widetilde{X}(\mathcal{O}_K) = \bigsqcup_{\mathfrak{c}\in\mathcal{C}^r} {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K)/\mathbb{G}^r_{\mathcal{O}_K}$, for a system $\mathcal{C}$ of representatives for the ideal classes of $\mathcal{O}_K$.

Then $U(K) = \bigsqcup_{\mathfrak{c}\in\mathcal{C}^r} \left( {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K) \cap \pi^{-1}(\psi^{-1}(U(K)))\right) /\mathbb{G}^r_{\mathcal{O}_K}$.

- Since $\widetilde{X}$ is split, smooth, with geometrically integral fibers, we can use it to verify Peyre's prediction on the constant $c$.

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