The "universal torsor method" for Manin's conjecture

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A singular del Pezzo surface

Let K be a number field, $\mathcal{O}_K$ its ring of integers.

$$S = \{x_0x_3 - x_2x_4 = x_0x_1 + x_1x_3 + x_2^2 = 0\} \subseteq \mathbb{P}^4_K$$

is a del Pezzo surface of degree 4 with singularities $A_3 + A_1$.

Its associated anticanonical height is

$$H : S(K) \to \mathbb{R}_{\geq 0}, \quad x = (x_0 : \ldots : x_4) \mapsto \prod_{\nu \in \Omega_K} \sup_{i=0,\ldots,4} |x_i|_\nu.$$

Let $U = S \smallsetminus (\{x_0 = x_1 = x_2 = 0\} \cup \{x_0 = x_2 = x_3 = 0\} \cup \{x_1 = x_2 = x_3 = 0\})$.

Let $X \to S$ be a minimal desingularization. Then $\mathrm{Pic}(X) \cong \mathbb{Z}^6$.

Manin's conjecture for S

Manin's conjecture for S

$$\#\{x \in U(K) : H(x) \le B\} \sim cB(\log B)^5, \quad B \to \infty,$$

where the constant c depends on X, K and H as predicted by E. Peyre.

Universal torsor method

- Parameterize $U(K)$ by integral points on a universal torsor Y of X.
- Lift the height function to Y
  and count lattice points of bounded height.

Torsors

Definition

Let S be a scheme, X an S-scheme and G an abelian S-group scheme. An X-torsor under G is
$$\begin{array}{ccc} G \circlearrowright Y & \xrightarrow{\pi} & X \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$
where π is *fppf* and Y is locally isomorphic to $G \times_S X$ with its action.

Parameterization property

$$X(S) = \bigsqcup_{\mathfrak{c} \in H^1_{fppf}(S,G)} {}_{\mathfrak{c}}Y(S)/G(S)$$

where ${}_{\mathfrak{c}}Y \to X$ is a $\mathfrak{c}$-twist of Y.

Universal torsor for X

Let $D_1, \dots, D_6$ be a basis of $\mathrm{Pic}(X) \cong \mathbb{Z}^6$.
The Cox ring of X is

$$\begin{aligned}\mathrm{Cox}(X) := \bigoplus_{(n_1,\dots,n_6)\in\mathbb{Z}^6} H^0(X, \mathcal{O}_X(n_1 D_1 + \cdots + n_r D_r)) = \\ = K[\eta_1, \dots, \eta_9]/(\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7).\end{aligned}$$

The universal torsor of X is an open subset of the spectrum of $\mathrm{Cox}(X)$:

$$\begin{aligned}Y := \mathrm{Spec}_X \left(\bigoplus_{(n_1,\dots,n_6)\in\mathbb{Z}^6} \mathcal{O}_X(n_1 D_1 + \cdots + n_r D_r) \right) = \\ = \{\eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7\} \smallsetminus \{f_1, \dots, f_m\} \subseteq \mathbb{A}^9,\end{aligned}$$

where $f_1, \dots, f_m$ are monomials.

Integral models

$$
\begin{array}{ccccccccc}
K & & \mathbb{A}^N_K & \supset & Y & \xrightarrow{\pi} & X & \longrightarrow S & \supset \; U \\
\Big| & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_K & & \mathbb{A}^N_{\mathcal{O}_K} & \supset & \widetilde{Y} & \xrightarrow{\tilde{\pi}} & \widetilde{X} & &
\end{array}
$$

- $\widetilde{X}$ is projective and smooth over $\mathcal{O}_K$;
- $\tilde{\pi}$ is a universal torsor of $\widetilde{X}$.

Hence,

$$X(K) = \widetilde{X}(\mathcal{O}_K) = \bigsqcup_{\mathfrak{c} \in H^1_{ét}(\mathcal{O}_K, \mathbb{G}_m^6)} {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K)/\mathbb{G}_m^6(\mathcal{O}_K),$$

where ${}_{\mathfrak{c}}Y$ is the $\mathfrak{c}$-twist of Y.

Parameterization lemma

Since $H^1_{\acute{e}t}(\mathcal{O}_K, \mathbb{G}_m^6) = Cl(\mathcal{O}_K)^6$
and $\mathbb{G}_m^6(\mathcal{O}_K) = (\mathcal{O}_K^\times)^6$,

$$X(K) = \bigsqcup_{\mathfrak{c} \in \mathcal{C}^6} {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K)/(\mathcal{O}_K^\times)^6$$

for a system $\mathcal{C}$ of representatives for the ideal classes of $\mathcal{O}_K$.
Thus,

$$\#\{x \in U(K) : H(x) \leq B\} =$$
$$= \frac{1}{(\#\mu_K)^6} \sum_{\mathfrak{c} \in \mathcal{C}^6} \#\left(\{y \in {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K) \cap U'(K) : H(\pi(y)) \leq B\}/U_K^6\right)$$

where U' is the preimage of U in Y, and $\mathcal{O}_K^\times = \mu_K \times U_K$.

Lifting of the height

If $y = (y_1, \dots, y_9) \in {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K)$, then

$$H(\pi(y)) = \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{c}^D)} \prod_{\nu \in \Omega_\infty} \sup_{i=0,\dots,4} |h_i(y)|_\nu,$$

where D is the anticanonical divisor of X and $h_0, \dots, h_4$ are monomials.

Let $H'(y) := \prod_{\nu \in \Omega_\infty} \sup_{i=0,\dots,4} |h_i(y)|_\nu$.
Then

$$\#\{x \in U(K) : H(x) \leq B\} =$$
$$= \frac{1}{(\#\mu_K)^6} \sum_{\mathfrak{c} \in \mathcal{C}^r} \#\left(\{y \in {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K) \cap U'(K) : H'(y) \leq N_{K/\mathbb{Q}}(\mathfrak{c}^D) B\} / U_K^6\right)$$

$$
\begin{array}{ccccccccccc}
K & & K^9 & \supset & Y(K) & \xrightarrow{\pi} & X(K) & \longrightarrow & S(K) & \supset & U(K) \\
\Big| & & \cup & & \cup & & \parallel & & & & \\
\mathcal{O}_K & & \bigoplus_{j=1}^{9} \mathfrak{c}^{\deg \eta_j} & \supset & {}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K) & \xrightarrow{\tilde{\pi}} & \widetilde{X}(\mathcal{O}_K) & & & &
\end{array}
$$

Let $g := \eta_1\eta_9 + \eta_2\eta_8 + \eta_3\eta_4^2\eta_5^3\eta_7$.
Then

$$Y = \{g = 0\} \smallsetminus \{f_1 = \cdots = f_m = 0\} \subseteq \mathbb{A}_K^9,$$

$$
\begin{aligned}
Y(K) &= \{y \in K^9 : g(y) = 0,\ \exists i \text{ s.t. } f_i(y) \neq 0\} = \\
&= \{y \in K^9 : g(y) = 0,\ f_1(y)K + \cdots + f_m(y)K = K\},
\end{aligned}
$$

$${}_{\mathfrak{c}}\widetilde{Y}(\mathcal{O}_K) = \{y \in \bigoplus_{j=1}^{9} \mathfrak{c}^{\deg \eta_j} : g(y) = 0,\ f_1(y)\mathfrak{c}^{-\deg f_1} + \cdots + f_m(y)\mathfrak{c}^{-\deg f_m} = \mathcal{O}_K\},$$

$$\widetilde{Y}(\mathcal{O}_K) = \{y \in \mathcal{O}_K^9 : g(y) = 0,\ f_1(y)\mathcal{O}_K + \cdots + f_m(y)\mathcal{O}_K = \mathcal{O}_K\}.$$

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