Generalized Cox rings over arbitary fields

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What are Cox rings?

1995: Cox introduced homogeneous coordinate rings of toric varieties. E.g.

$$\operatorname{Spec} \mathbb{C}[x_0, \dots, x_n] \supseteq \mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\} \to \mathbb{P}^n_{\mathbb{C}} = (\mathbb{A}^{n+1}_{\mathbb{C}} \smallsetminus \{0\}) // \mathbb{G}_m.$$

2000: Hu and Keel defined Cox rings of a smooth projective variety X with $\operatorname{Pic}(X) \cong \mathbb{Z}^r$ as the $\operatorname{Pic}(X)$-graded rings

$$\bigoplus_{[D] \in \operatorname{Pic}(X)} H^0(X, \mathcal{O}_X(D))$$

with multiplication induced by the choice of a basis of $\operatorname{Pic}(X)$.

$\operatorname{Cl}(X)$-graded Cox rings: EKW 2004, Hausen 2008, ADHL 2015
$\operatorname{Pic}(X)$-graded Cox rings: Berchtold-Hausen 2003

Why Cox rings?

(MMP): a variety X is a Mori dream space $\iff$ its Cox ring R is finitely generated as k-algebra, in this case

$$X = (\operatorname{Spec} R \setminus V(I))//H_{ns},$$

where H_{ns} is the Neron-Severi quasitorus and I is called irrelevant ideal.

If R is $\operatorname{Pic}(X)$-graded, then $\operatorname{Spec} R \setminus V(I)$ is a universal torsor of X.

Universal torsors have been introduced by Colliot-Thélène and Sansuc to study the arithmetic of certain varieties over number fields, in particular the existence of rational points, as

$$X(k) = \bigsqcup_{\alpha \in H^1(k, H_{ns})} \pi^\alpha(Y^\alpha(k)),$$

where $\pi^\alpha : Y^\alpha \to X$ are twisted universal torsors.

Universal torsors and Cox rings are also used to study the distribution of rational points on (quasi)-Fano varieties with respect to anticanonical height functions

$$H : X(k) \to \mathbb{R}_{\geq 0}.$$

Conjecture (Manin, 1989):
If k is a number field and $X(k)$ is dense in X, then there is an open subset $U \subseteq X$ such that

$$\#\{x \in U(k) : H(x) \leq B\} \sim CB(\log B)^{r-1},$$

where $C > 0$ and $r = \operatorname{rk} \operatorname{Pic}(X)$.

Universal torsors have been used mostly for split varieties (i.e. with trivial Galois action on $\operatorname{Pic}(X_{\overline{k}})$). Some proofs of Manin's conjecture for certain non-split varieties use other torsors.

Torsors under quasitori

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$.

Let G be a quasitorus (=smooth group of multiplicative type). Then X-torsors under G are classified by $H^1_{\acute{e}t}(X, G_X)$.

(Colliot-Thélène – Sansuc) There is an exact sequence

$$0 \longrightarrow H^1_{\acute{e}t}(k, G) \longrightarrow H^1_{\acute{e}t}(X, G_X) \xrightarrow{type} \mathrm{Hom}_{\mathfrak{g}-\mathrm{mod}}(\widehat{G}, \mathrm{Pic}(X_{\overline{k}})).$$

Universal torsors of X are torsors under quasitori of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$.

Question: what are "Cox rings" for torsors under arbitrary quasitori over arbitrary fields?

Quasitori

(Grothendieck, SGA3) There is an antiequivalence of categories

$$\begin{aligned} \{\text{Quasitori over } k\} &\longrightarrow \left\{\begin{matrix}\text{Finitely generated } \mathfrak{g}\text{-modules} \\ \text{(with no } p\text{-torsion if } \mathrm{char}(k) = p > 0)\end{matrix}\right\} =: \mathcal{M} \\ G &\longmapsto \widehat{G} := \mathrm{Hom}_{\overline{k}}(G_{\overline{k}}, \mathbb{G}_{m,\overline{k}}) \\ \mathrm{Spec}\,\overline{k}[M]^{\mathfrak{g}} =: \widehat{M} &\longleftarrow M. \end{aligned}$$

For $M \in \mathcal{M}$, consider the exact sequence

$$0 \longrightarrow H^1_{ét}(k, \widehat{M}) \longrightarrow H^1_{ét}(X, \widehat{M}_X) \xrightarrow{type} \mathrm{Hom}_{\mathfrak{g}-\mathrm{mod}}(M, \mathrm{Pic}(X_{\overline{k}})).$$

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$,
$\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$

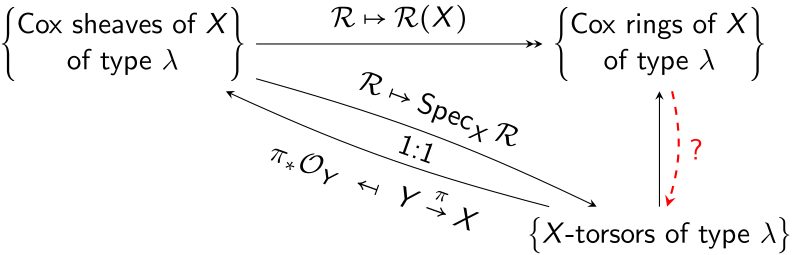
Definition

($k = \overline{k}$) A generalized Cox sheaf of X of type λ is a structure of
ring
M-graded $\mathcal{O}_X$-algebra on $\bigoplus_{m \in M} \mathcal{O}_X(D_m)$, for Cartier
k- $\bigoplus_{m \in M} H^0(X, \mathcal{O}_X(D_m))$
divisors D_m such that $[D_m] = \lambda(m)$ with multiplication compatible with sum of divisors.

(k arbitrary) A generalized Cox sheaf of X of type λ is an $\mathcal{O}_X$-algebra $\mathcal{R}$
ring k-
such that $\mathcal{R} \otimes_k \overline{k}$ has a structure of generalized Cox sheaf of
ring
$X_{\overline{k}}$ of type λ which is compatible with the induced $\mathfrak{g}$-action.

Classification theorem

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$



- $\mathcal{R} \mapsto \mathcal{R}(X)$ is ess. inj. if $M = \langle m \in M : \lambda(m)$ effective $\rangle =: M_{\mathrm{eff}}$.
- The automorphism group of a Cox sheaf of $X_{\overline{k}}$ of type λ is $\widehat{M}(\overline{k}) = Hom(M, \overline{k}^\times)$. For a Cox ring of type λ it is $\widehat{M_{\mathrm{eff}}}(\overline{k})$.
- Isomorphism classes of Cox sheaves of X of type λ are classified by $H^1_{ét}(k, \widehat{M})$.

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$

Proposition ($k = \overline{k}$)

Let $\mathcal{R}$ be a Cox sheaf of X of type λ. If $\mathcal{R}(X)$ is finitely generated as k-algebra, and $\exists\ f_1, \dots, f_t \in \mathcal{R}(X)$ nonzero and homogeneous such that $X \smallsetminus \mathrm{Supp}(\mathrm{div}(f_i))$ are affine and cover X, then

$$\mathrm{Spec}_X \mathcal{R} \cong \mathrm{Spec}\, \mathcal{R}(X) \smallsetminus V(f_1, \dots, f_t).$$

Remarks

- $f_1, \dots, f_t$ as above exist if $\lambda(M)$ contains an ample divisor class.
- If X and $\mathcal{R}$ are defined over a nonclosed k, the isomorphism above is $\mathfrak{g}$-equivariant.

Pullback

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$

Proposition

($k = \overline{k}$) Let $\mathcal{R}$ be a Cox sheaf/ring of X of type λ, $M' \in \mathcal{M}$ and $\varphi : M' \to M$ a homomorphism. The pullback of $\mathcal{R} = \bigoplus_{m \in M} \mathcal{R}_m$ under φ

$$\varphi^* \mathcal{R} := \bigoplus_{m' \in M'} \mathcal{R}_{\varphi(m')}$$

is a Cox sheaf/ring of X of type $\lambda \circ \varphi$.

Remark

- If X and $\mathcal{R}$ are defined over a nonclosed k and φ is a morphism of $\mathfrak{g}$-modules, the pullback is $\mathfrak{g}$-equivariant.
- The pullback of a finitely generated generalized Cox ring is finitely generated.

Application: Châtelet surfaces

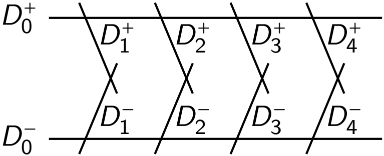
Application:

Generalized Cox rings of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ explain parameterizations in proofs of Manin's conjecture for some non-split varieties.

Example: Let X be a Chatelet surface:

$$x^2 + y^2 = P(z).$$

The classes of the divisors



generate $\mathrm{Pic}(X_{\overline{k}})$ and a Cox ring of $X_{\overline{k}}$ of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$ is

$$\overline{k}[\eta_0^\pm, \dots, \eta_4^\pm]/(\Delta_{i,j}\eta_l^+\eta_l^- + \Delta_{j,l}\eta_i^+\eta_i^- + \Delta_{l,i}\eta_j^+\eta_j^-)_{1\le i<j<l\le 4}$$

Theorem (de la Bretèche - Browning - Peyre)

Manin's conjecture holds for

$$x^2 + y^2 = L_1(z)L_2(z)L_3(z)L_4(z).$$

The proof uses both torsors of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ and of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$.

Theorem (Destagnol)

Manin's conjecture holds for

$$x^2 + y^2 = L_1(z)L_2(z)Q(z).$$

The proof uses torsors of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ and of type

$$\langle \bar{D}_0^{\pm},\ \bar{D}_1^{\pm},\ \bar{D}_2^{\pm},\ \bar{D}_3^{+} + \bar{D}_4^{+},\ \bar{D}_3^{-} + \bar{D}_4^{-} \rangle \subseteq \mathrm{Pic}(X_{\overline{k}}).$$

Thank you for your attention.