Generalized Cox rings over nonclosed fields

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Introduction: Cox rings and universal torsors

What are Cox rings?

The Cox ring of a variety X over $\mathbb{C}$ is a $\mathrm{Pic}(X)$-graded $\mathbb{C}$-algebra

$$\bigoplus_{\mathrm{Pic}(X)} H^0(X, \mathcal{O}_X(D)).$$

- Introduced by Hu and Keel in 2000 to study Mori Dream Spaces.
- Precursory work of Cox in 1995 on homogeneous coordinate rings of toric varieties.

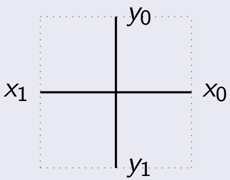
Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Picard group: $\mathbb{Z}^2$,

Cox ring: $\mathbb{C}[x_0, x_1, y_0, y_1]$,

$\deg x_0 = \deg x_1 = (1, 0)$,

$\deg y_0 = \deg y_1 = (0, 1)$.



Why Cox rings

A variety X is a Mori Dream Space $\iff$ its Cox ring is finitely generated.

If X has a finitely generated Cox ring R,

$$\operatorname{Spec} R \overset{\text{open}}{\supset} Y \xrightarrow[\text{univ. torsor}]{//H} X$$

- $Y = \operatorname{Spec}_X \mathcal{R}$, where $\mathcal{R}$ is a Cox sheaf of X.
- X-torsor under H: étale locally $X \times H$.
- Universal torsors are special torsors under quasitori.

Torsors under quasitori

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$.

The quasitorus associated to a $\mathfrak{g}$-module M is $\widehat{M} := \mathrm{Spec}\,\overline{k}[M]^{\mathfrak{g}}$.

X-torsors under $\widehat{M}$ are classified by $H^1_{ét}(X, \widehat{M})$:

$$0 \longrightarrow H^1_{ét}(k, \widehat{M}) \longrightarrow H^1_{ét}(X, \widehat{M}) \xrightarrow{type} \mathrm{Hom}_{\mathfrak{g}-\mathrm{mod}}(M, \mathrm{Pic}(X_{\overline{k}})).$$

Universal torsors of X are torsors with $M = \mathrm{Pic}(X_{\overline{k}})$ and type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{k}})}$.

Question: what are "Cox rings" for torsors of arbitrary type over arbitrary fields?

Generalized Cox sheaves and Cox rings

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$

Definition

($k = \overline{k}$) A generalized Cox sheaf of X of type λ is an M-graded

ring

$\mathcal{O}_X$-algebra $\bigoplus_{m\in M} \mathcal{O}_X(D_m)$, where $[D_m] = \lambda(m)$

k- $\bigoplus_{m\in M} H^0(X, \mathcal{O}_X(D_m))$

$\forall m \in M$, and the multiplication is compatible with the sum of divisors.

(k arbitrary) A generalized Cox sheaf of X of type λ is an $\mathcal{O}_X$-algebra $\mathcal{R}$

ring k-

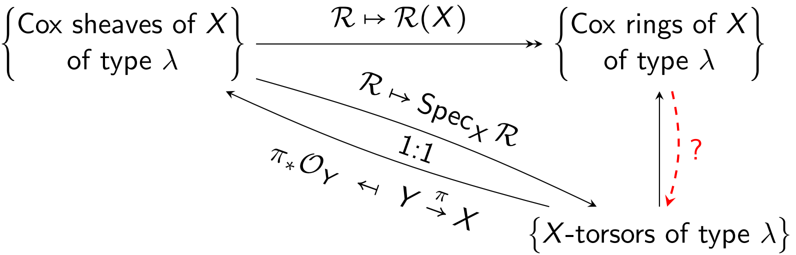
such that $\mathcal{R} \otimes_k \overline{k}$ has a structure of generalized Cox sheaf of

ring

$X_{\overline{k}}$ of type λ which is compatible with the induced $\mathfrak{g}$-action.

Classification theorem

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$



- $\mathcal{R} \mapsto \mathcal{R}(X)$ is ess. inj. if $M = \langle m \in M : \lambda(m) \text{ effective}\rangle =: M_{\mathrm{eff}}$.
- The automorphism group of a Cox sheaf of $X_{\overline{k}}$ of type λ is $\widehat{M}(\overline{k}) = Hom(M, \overline{k}^\times)$. For a Cox ring of type λ it is $\widehat{M_{\mathrm{eff}}}(\overline{k})$.
- Isomorphism classes of Cox sheaves of X of type λ are classified by $H^1_{\acute{e}t}(k, \widehat{M})$.

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$

Proposition ($k = \overline{k}$)

Let $\mathcal{R}$ be a Cox sheaf of X of type λ. If $\mathcal{R}(X)$ is finitely generated as k-algebra, and $\exists\ f_1, \dots, f_t \in \mathcal{R}(X)$ nonzero and homogeneous such that $X \smallsetminus \mathrm{Supp}(\mathrm{div}(f_i))$ are affine and cover X, then

$$\mathrm{Spec}_X \mathcal{R} \cong \mathrm{Spec}\, \mathcal{R}(X) \smallsetminus V(f_1, \dots, f_t).$$

Remarks

- If $\lambda(M)$ contains an ample divisor class, $f_1, \dots, f_t$ as above exist.
- If X and $\mathcal{R}$ are defined over a nonclosed k, the isomorphism above is $\mathfrak{g}$-equivariant.

Pullback and computations

Pullback

k field, $\overline{k}$ sep. closure, $\mathfrak{g} = \mathrm{Gal}(\overline{k}/k)$, X geom. integral k-variety, $\overline{k}[X_{\overline{k}}]^\times = \overline{k}^\times$, $\lambda : M \to \mathrm{Pic}(X_{\overline{k}})$

Proposition ($k = \overline{k}$)

Let $\mathcal{R}$ be a Cox sheaf/ring of X of type λ, and $\varphi : M' \to M$ a group homomorphism. The pullback of $\mathcal{R} = \bigoplus_{m \in M} \mathcal{R}_m$ under φ

$$\varphi^*\mathcal{R} := \bigoplus_{m' \in M'} \mathcal{R}_{\varphi(m')}$$

is a Cox sheaf/ring of X of type $\lambda \circ \varphi$.

Remark

- The pullback is $\mathfrak{g}$-equivariant.
- With $\varphi = \lambda$, a Cox ring of type λ is the pullback of a Cox ring of type $\mathrm{id}_{\mathrm{Pic}(X)}$.
- The pullback preserves finite generation.

Trivial example

$X = \mathbb{P}^1 \times \mathbb{P}^1$

Picard group: $\mathbb{Z}^2$,

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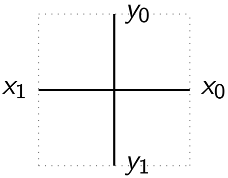


A generalized Cox ring of X of type

$$\lambda : \mathbb{Z} \to \mathbb{Z}^2, \qquad a \mapsto (a, -a),$$

is

$$\lambda^* \mathbb{C}[x_0, x_1, y_0, y_1] = \bigoplus_{a \in \mathbb{Z}} \mathbb{C}[x_0, x_1, y_0, y_1]_{(a,-a)} = \mathbb{C}.$$

In this case, $M = \mathbb{Z}$ and $M_{\text{eff}} = \{0\}$.

Computations via pullback ($k = \overline{k}$)

- Over $\mathbb{C}$, finitely generated Cox rings of type $\mathrm{id}_{\mathrm{Pic}(X)}$ have been computed for many varieties. See work of Altmann, Batyrev, Berchtold, Castravet, Derenthal, Hassett, Hausen, Keicher, Laface, Popov, Testa, Tevelev, Tschinkel, Várilly-Alvarado, Velasco,...
- Every generalized Cox ring is the pullback of a Cox ring of type $\mathrm{id}_{\mathrm{Pic}(X)}$ under the type map $\lambda : M \to \mathrm{Pic}(X)$.
- The pullback preserves finite generation.

General strategy (λ injective)

$k = \overline{k}$ sep. closed field, X integral k-variety, $k[X]^\times = k^\times$, $\lambda : M \hookrightarrow \mathrm{Pic}(X)$

Generators

If

- $R = k[\eta_1, \dots, \eta_N]/I$ is a Cox ring of type $\mathrm{id}_{\mathrm{Pic}(X)}$,
- $\eta_1, \dots, \eta_N$ homogeneous of degrees $[D_1], \dots, [D_N]$,

the Cox ring $\lambda^* R = \bigoplus_{m \in M} \mathcal{R}_{\lambda(m)}$ of type λ is generated by the monomials

$$\eta_1^{a_1} \dots \eta_N^{a_N} \quad \text{s. t.} \quad [a_1 D_1 + \dots + a_N D_N] \in \lambda(M).$$

If $\mathrm{Pic}(X)$ is free, finding the generators is the same as solving a system of integral linear equations in $\mathbb{Z}_{\geq 0}$.

General strategy (λ injective)

$k = \overline{k}$ sep. closed field, X integral k-variety, $k[X]^\times = k^\times$, $\lambda : M \hookrightarrow \mathrm{Pic}(X)$

Relations

If

- $R = k[\eta_1, \dots, \eta_N]/I$ is a Cox ring of type $\mathrm{id}_{\mathrm{Pic}(X)}$,
- $\xi_1, \dots, \xi_{N'} \in R$ are the generators of $\lambda^* R$

it remains to compute the kernel of

$$k[\xi_1, \dots, \xi_{N'}] \to R.$$

Fact: If M contains an ample divisor class,

$$\dim \lambda^* R = \dim X + \operatorname{rank} M = \dim R + \operatorname{rank} M - \operatorname{rank} \mathrm{Pic}(X).$$

$\to$ If $N' = \dim \lambda^* R + 1$, it is enough to find one irreducible relation.

An arithmetic application

Arithmetic motivation

Universal torsors and Cox rings are used to study the distribution of rational points on (quasi)-Fano varieties with respect to anticanonical height functions

$$H : X(k) \to \mathbb{R}_{\geq 0}.$$

Conjecture (Manin, 1989):
If k is a number field and $X(k)$ is dense in X, then there is an open subset $U \subseteq X$ such that

$$\#\{x \in U(k) : H(x) \leq B\} \sim CB(\log B)^{r-1},$$

where $C > 0$ and $r = \operatorname{rk}\operatorname{Pic}(X)$.

Universal torsors have been used mostly for split varieties (i.e. with trivial Galois action on $\operatorname{Pic}(X_{\overline{k}})$). Some proofs of Manin's conjecture for certain non-split varieties use other torsors of injective type.

Application:

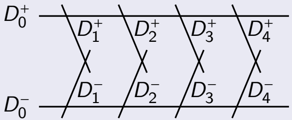
Generalized Cox rings of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ explain parameterizations in proofs of Manin's conjecture for some non-split varieties.

Example: Châtelet surfaces

Let X be a Châtelet surface:

$$x^2 + y^2 = P(z).$$

The Picard group of $X_{\overline{\mathbb{Q}}}$ is generated by the classes of the divisors



a Cox ring of $X_{\overline{\mathbb{Q}}}$ of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{\mathbb{Q}}})}$ has 10 generators and 2 quadratic relations.

$$Cox(X_{\overline{\mathbb{Q}}}) = \overline{\mathbb{Q}}[\eta_0^{\pm}, \dots, \eta_4^{\pm}]/(\Delta_{i,j}\eta_l^+\eta_l^- + \Delta_{j,l}\eta_i^+\eta_i^- + \Delta_{l,i}\eta_j^+\eta_j^-)_{1\le i<j<l\le 4}$$

Theorem (de la Bretèche - Browning - Peyre, 2012)

Manin's conjecture holds for

$$x^2 + y^2 = L_1(z)L_2(z)L_3(z)L_4(z).$$

The proof uses both torsors of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{\mathbb{Q}}})$ and of type $\mathrm{id}_{\mathrm{Pic}(X_{\overline{\mathbb{Q}}})}$.

$\mathrm{id}_{\mathrm{Pic}(X_{\overline{\mathbb{Q}}})}$: $\mathbb{Q}[x_0, y_0, \dots, x_4, y_4]/(\Delta_{i,j}(x_l^2 + y_l^2) + \Delta_{j,l}(x_i^2 + y_i^2) + \Delta_{l,i}(x_j^2 + y_j^2))_{1\le i<j<l\le 4}$
via $x_i = \eta_i^+ + \mathrm{i}\eta_i^-$, $y_i = \eta_i^+ - \mathrm{i}\eta_i^-$.

$\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{\mathbb{Q}}})$: 5 generators $x + \mathrm{i}y = \eta_0^+ \prod_{j=0}^4 \eta_j^+$, $x - \mathrm{i}y = \eta_0^- \prod_{j=0}^4 \eta_j^-$,
$t = \eta_0^+\eta_0^-$, $L_j(u, v) = \eta_j^+\eta_j^-$, $j \in \{1, \dots, 4\}$,

1 relation $x^2 + y^2 = t^2 L_1(u, v)L_2(u, v)L_3(u, v)L_4(u, v)$.

Theorem (Destagnol, 2015)

Manin's conjecture holds for

$$x^2 + y^2 = L_1(z)L_2(z)Q(z),$$

Q irreducible over $\mathbb{Q}(\mathrm{i})$.

The proof uses torsors of type $\mathrm{Pic}(X) \subseteq \mathrm{Pic}(X_{\overline{k}})$ and of type

$$\langle \bar{D}_0^{\pm},\ \bar{D}_1^{\pm},\ \bar{D}_2^{\pm},\ \bar{D}_3^{+} + \bar{D}_4^{+},\ \bar{D}_3^{-} + \bar{D}_4^{-} \rangle \subseteq \mathrm{Pic}(X_{\overline{k}}).$$

Thank you for your attention.