Campana points, a new number theoretic challenge

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Slogan

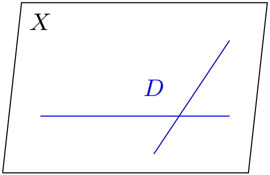
Given a pair

$$(X, D)$$

X: smooth proj. variety; D: reduced snc divisor

over $\mathbb{Q}$ with integral model $(\mathscr{X}, \mathscr{D})$.

eg:



- Campana points interpolate between integral points on $\mathscr{X} \smallsetminus \mathscr{D}$ and rational points on X.
- Campana points are rational points on X with special “tangency” conditions with respect to D.

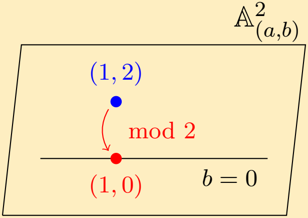
Rational points vs integral points

Rational points and integral points on projective models coincide.

If $\mathscr{D} \subsetneq \mathscr{X}$ closed, $(\mathscr{X} \smallsetminus \mathscr{D})(\mathbb{Q}) \neq (\mathscr{X} \smallsetminus \mathscr{D})(\mathbb{Z})$.

Eg.: $(\mathscr{X}, \mathscr{D}) = (\mathbb{P}^1, \{\infty\})$

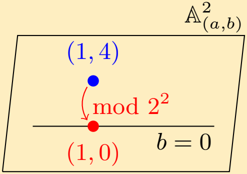
$$\begin{array}{l} \mathbb{P}^1(\mathbb{Q}) = \mathbb{A}^1(\mathbb{Q}) \cup \{\infty\} = \mathbb{Q} \cup \{\infty\} \\ \quad \| \\ \mathbb{P}^1(\mathbb{Z}) \supsetneq \mathbb{A}^1(\mathbb{Z}) \cup \{\infty\} = \mathbb{Z} \cup \{\infty\} \end{array}$$



A point x is integral with respect to $\mathscr{D}$ if modulo every prime $x \notin \mathscr{D}$.

Approximating integral points by Campana points

Eg.: $(\mathscr{X}, \mathscr{D}) = (\mathbb{P}^1, \{\infty\})$



$$\mathbb{Q} = \{\tfrac{a}{b} : a,b \in \mathbb{Z}, b \neq 0, \gcd(a,b)=1\}$$

$$\cup\!|$$

$$\{p \mid b \Rightarrow p^2 \mid b\}$$

$$\cup\!|$$

$$\{p \mid b \Rightarrow p^3 \mid b\} \supseteq \dots \supseteq \bigcap_{m \geq 1} \{p \mid b \Rightarrow p^m \mid b\}$$

$$\|$$

$$\{b = \pm 1\} = \mathbb{Z}$$

A point x is Campana with multiplicity m if for every prime p,

$$x \in \mathscr{D} \text{ modulo } p \quad \Rightarrow \quad x \in \mathscr{D} \text{ modulo } p^m.$$

The Campana condition

Definition: An integer x is m-full if $x \neq 0$ and $p \mid x \Rightarrow p^m \mid x$.

Campana condition

Given:

$D_1, \dots, D_n$ components of D,

$m_1, \dots, m_n \in \mathbb{Z}_{>0}$, $\mathbf{m} = (m_1, \dots, m_n)$,

a point $x \in \mathscr{X}(\mathbb{Z})$ is Campana wrt $(\mathscr{D}, \mathbf{m})$ if

$$f_i(x) \text{ is } m_i\text{-full}$$

for every i and every local equation f_i of $\mathscr{D}_i$ around x.

Example: $\mathscr{X} = \mathbb{P}^1$, $\mathscr{D} = \{0, 1, \infty\}$, $\mathbf{m} = (m_1, m_2, m_3)$

$$\{a, b \in \mathbb{Z} \text{ coprime} : a \text{ is } m_1\text{-full}, a - b \text{ is } m_2\text{-full}, b \text{ is } m_3\text{-full}\}$$

Question:

If X is a curve and $X(\mathbb{Q})$ is infinite, under what conditions on $(\mathscr{D}, \mathbf{m})$ is the set of Campana points finite?

Example: $\mathscr{X} = \mathbb{P}^1$, $\mathscr{D} = \{0, 1, \infty\}$, $\mathbf{m} = (m_1, m_2, m_3)$

$$\{a, b \in \mathbb{Z} \text{ coprime} : a \text{ is } m_1\text{-full}, a - b \text{ is } m_2\text{-full}, b \text{ is } m_3\text{-full}\}$$

For $m_1 = m_2 = m_3 \geq 4$ finiteness follows from the abc-conjecture.

Orbifold Mordell conjecture (Campana 2005)

If $K_X + \sum_{i=1}^n (1 - 1/m_i) D_i$ is ample, the set of Campana points is finite.

Orbifold Mordell follows from the abc-conjecture.

Theorem (Campana 2005, Kebekus–Pereira–Smeets 2019)

Orbifold Mordell holds over function fields of curves.

Manin-type conjecture

Conjecture (P.–Smeets–Tanimoto–Várilly-Alvarado 2019)

If $K/\mathbb{Q}$ finite,
$-(K_X + \sum_{i=1}^n (1 - 1/m_i)D_i)$ ample,
Campana points are Zariski dense in X,
L adelically metrized ample line bundle,

then $\exists$ thin set $Z \subset X(K)$ such that

$$\#\{\text{Campana points } \notin Z : H_L(x) \leq B\} \sim cB^a(\log B)^{b-1},$$

with $c > 0$ and

$a = \inf\{t \in \mathbb{R} : tL + K_X + \sum_{i=1}^n (1 - 1/m_i)D_i \text{ effective}\}$,
$b =$ dimension of the minimal face of $\mathrm{Eff}(X)$ containing $aL + K_X + \sum_{i=1}^n (1 - 1/m_i)D_i$.

Results

Example: $\mathscr{X} = \mathbb{P}^1$, $\mathscr{D} = \{0, 1, \infty\}$, $\mathbf{m} = (2, 2, 2)$

$$\#\{a, b \in \mathbb{Z} \text{ coprime} : a, b, a-b \text{ is } 2\text{-full}, |a|, |b| \leq B\} \overset{?}{\sim} cB^{1/2}$$

Open problem.

(Browning–Van Valckenborgh 2012): $\gg B^{1/2}$, $\ll B^{3/5}(\log B)^{12}$.

Higher dimension: circle method

Let $\mathscr{X} = \mathbb{P}^{n-2}$, $\mathscr{D}_i = \{x_i = 0\}, 1 \leq i \leq n-1$, $\mathscr{D}_n = \{\sum_{i=1}^{n-1} x_i = 0\}$.

(Van Valckenborgh 2012): Manin-type conjecture holds for $n \geq 5$, $m_1 = \cdots = m_n = 2$.

(Browning–Yamagishi 2019): Manin-type conjecture holds if $m_1, \ldots, m_n \geq 2$ and $\sum_{\substack{1 \leq i \leq n \\ i \neq j}} \frac{1}{m_i(m_i+1)} \geq 1$ for some $j \in \{1, \ldots, n\}$.

Theorem (P.–Smeets–Tanimoto–Várilly-Alvarado 2019)

Manin-type conjecture holds for smooth compactifications of vector groups with equivariant boundary over arbitrary number fields.

- Proof: harmonic analysis of the height zeta function.

$$Z(s) = \sum_{x \in \text{Campana points}} H(x)^{-s}$$

Meromorphic continuation via Poisson summation formula

$$Z(s) = \sum_{x \in G(k)} H(x)^{-s} \delta_{\mathbf{m}}(x) = \sum_{x \in G(k)} \widehat{H}_{\mathbf{m}}(x, s)$$

+ Tauberian theorem.

Theorem (P.–Schindler 2020)

Manin-type conjecture holds for many split toric varieties with equivariant boundary over $\mathbb{Q}$ and equipped with the log-anticanonical height.

- Proof: hyperbola method on split torsors.

$$\#\{x \in \underbrace{\Lambda}_{\text{lattice}} : \underbrace{\textcolor{blue}{\text{(constraints)}}}_{\text{independent of B}}, \underbrace{\textcolor{red}{\text{(height)}}}_{H(x)\leq B}\} = \sum_{x\in\Lambda,\textcolor{red}{\text{(height)}}} \textcolor{blue}{\bigstar}$$

Hyperbola method

Idea: separate (constraints) and (height) into two independent problems:

1. count lattice points with (constraints) and easy height $\rightsquigarrow \bigstar$
2. count $\bigstar$ with easy constraints and original (height).

Theorem (Xiao 2020)

Manin-type conjecture holds for biequivariant compactifications of the Heisenberg group with equivariant boundary over $\mathbb{Q}$.

Theorem (Streeter 2020)

E/K extension of number fields of degree d. Asymptotic formula for $\mathscr{X} = \mathbb{P}^{d-1}$, $\mathscr{D} = \{N_{E/K} = 0\}$, $m \geq 2$ coprime to d.

More open problems

Assume that $-(K_X + \sum_{i=1}^{n}(1 - 1/m_i)D_i)$ is ample.

- Do Campana points satisfy potential density?
- If Zariski dense, is the set of Campana points not thin?
- Do Campana points satisfy weak-weak approximation?