On the distribution of Campana points on Fano varieties

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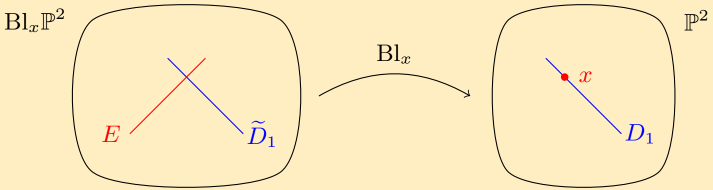
Setting:

- $F/\mathbb{Q}$ finite
- $G = \mathbb{G}_a^n$
- X smooth projective equivariant compactification of G
- $D = X \setminus G = \bigcup_{\alpha \in \mathcal{A}} D_\alpha$ snc divisor

Examples

$X_2 = \mathrm{Bl}_x \mathbb{P}^n$
$D_2 = \widetilde{D}_1 \cup E$

$X_1 = \mathbb{P}^n$
$D_1 = \{x_0 = 0\}$



Slogan

- Campana points interpolate between integral points and rational points on G.
- Campana points are rational points on X with special “tangency” conditions with respect to D.

Setting:

S finite set of places of F

$\mathscr{X}$ regular flat proper model of X over $\mathcal{O}_{F,S}$

$\mathscr{D}_\alpha \subseteq \mathscr{X}$ closure of D_α, $\forall \alpha \in \mathcal{A}$

$m_\alpha \in \mathbb{Z}_{\geq 1} \cup \{+\infty\}$, $\forall \alpha \in \mathcal{A}$

$D_{\mathbf{m}} := \sum_{\alpha \in \mathcal{A}} (1 - 1/m_\alpha) D_\alpha$, $\mathscr{D}_{\mathbf{m}} := \sum_{\alpha \in \mathcal{A}} (1 - 1/m_\alpha) \mathscr{D}_\alpha$

then $(X, D_{\mathbf{m}})$ is a Campana pair.

Campana points on $(\mathscr{X}, \mathscr{D}_{\mathbf{m}})$

A point $x \in G(F)$ is a Campana point on $(\mathscr{X}, \mathscr{D}_{\mathbf{m}})$ if

$$v_{\mathfrak{p}}(f_\alpha(x)) = 0 \qquad \text{or} \qquad v_{\mathfrak{p}}(f_\alpha(x)) \geq m_\alpha,$$

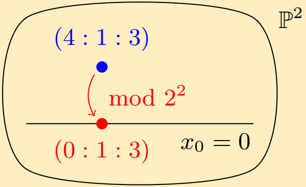
$\forall \alpha \in \mathcal{A}$, $\forall$ prime $\mathfrak{p}$ of $\mathcal{O}_{F,S}$, $\forall$ local equation f_α of $\mathscr{D}_\alpha$.

Example: $\mathscr{X} = \mathbb{P}^2$, $\mathscr{D}_m = (1-1/m)\{x_0 = 0\}$

A local equation at x is x_0/l,

l linear form such that $l(x) = 1$.

Eg.: $x_0/(x_0 - x_2)$ for $(4:1:3)$.



Let L be a big line bundle on X with a smooth adelic metrization induced by $\mathscr{X}$, and H_L the associated height function.

Theorem (P.–Smeets–Tanimoto–Várilly-Alvarado 2019)

Assume $m_\alpha < +\infty$ for all $\alpha \in \mathcal{A}$. Then $(X, D_{\mathbf{m}})$ is log Fano, the set of Campana points is Zariski dense in X, and

$$\#\{x \in G(F) \text{ Campana for } (\mathscr{X}, \mathscr{D}_m) : H_L(x) \leq B\} \sim cB^a(\log B)^{b-1},$$

with $c > 0$ and

$a = \inf\{t \in \mathbb{R} : tL + K_X + D_{\mathbf{m}} \text{ effective}\}$,

$b =$ dimension of the minimal face of $\mathrm{Eff}(X)$ containing $aL + K_X + D_{\mathbf{m}}$,

as long as $aL + K_X + D_{\mathbf{m}}$ is rigid (eg., if $L = -(K_X + D_{\mathbf{m}})$).

Recall: $a = \inf\{t \in \mathbb{R} : tL + K_X + D_{\mathbf{m}}$ effective$\}$,
$b =$ dimension of the minimal face of $\mathrm{Eff}(X)$ containing $aL + K_X + D_{\mathbf{m}}$.

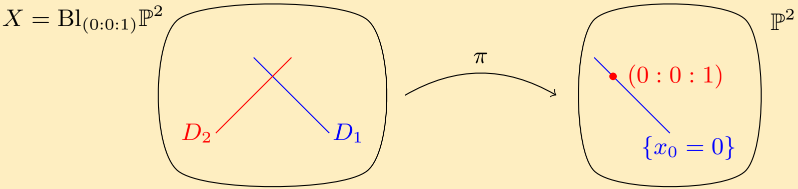
Facts:

- $\mathrm{Pic}(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z} D_\alpha, \quad \mathrm{Eff}(X) = \bigoplus_{\alpha \in \mathcal{A}} \mathbb{Z}_{\geq 0} D_\alpha,$
- $-K_X = \sum_{\alpha \in \mathcal{A}} \rho_\alpha D_\alpha$ with $\rho_\alpha \geq 2 \ \forall \alpha \in \mathcal{A}$,
- $L = \sum_{\alpha \in \mathcal{A}} \lambda_\alpha D_\alpha$ big $\Rightarrow \lambda_\alpha > 0 \ \forall \alpha \in \mathcal{A}$.

Then

$$a = \max_{\alpha \in \mathcal{A}} \frac{\rho_\alpha - (1 - \frac{1}{m_\alpha})}{\lambda_\alpha}, \qquad b = \# \left\{ \alpha \in \mathcal{A} : \frac{\rho_\alpha - (1 - \frac{1}{m_\alpha})}{\lambda_\alpha} = a \right\}.$$

Example



Let $G(F)_{\mathbf{m}} = \{x \in G(F)$ Campana for $(\mathscr{X}, \mathscr{D}_{\mathbf{m}})\}$, then

$$\pi(G(F)_{\mathbf{m}}) = \{(z_0, z_1, z_2) \in \mathbb{Z}^3 : \gcd(z_0, z_1, z_2) = 1, z_0 > 0, \text{ and } \forall p \; v_p(\gcd(z_0, z_1)) = 0 \text{ or } v_p(\gcd(z_0, z_1)) \geq m_1, \; v_p(z_0/\gcd(z_0, z_1)) = 0 \text{ or } v_p(z_0/\gcd(z_0, z_1)) \geq m_2\}.$$

If $L = \mathcal{O}_{\mathbb{P}^2}$, then

$$\#\{x \in \pi(G(F)_{\mathbf{m}}) : H_{\mathbb{P}^2}(x) \leq B\} \sim cB^{2+1/m_2}.$$

Proof of theorem

Let $G(F)_{\mathbf{m}} = \{x \in G(F)$ Campana for $(\mathscr{X}, \mathscr{D}_{\mathbf{m}})\}$.

Height zeta function: $$Z_{\mathbf{m}}(s) := \sum_{x \in G(F)_{\mathbf{m}}} H_L(x)^{-s}, \quad s \in \mathbb{C}.$$

Tauberian theorem

If $\sum_{x \in G(F)_{\mathbf{m}}} H_L(x)^{-s}$ is absolutely convergent for $\Re(s) > a > 0$, and

$$Z_{\mathbf{m}}(s) = (s-a)^{-b} g(s) + h(s)$$

with g and h holomorphic for $\Re(s) \geq a$, $g(a) \neq 0$, then

$$\#\{x \in G(F)_{\mathbf{m}} : H_L(x) \leq B\} \sim \frac{g(a)}{a(b-1)!} B^a (\log B)^{b-1}.$$

$\rightsquigarrow$ Suffice to study the **rightmost pole** of $Z_{\mathbf{m}}(s)$.

Poisson summation for $G = \mathbb{G}_a^n$

Recall: $Z_{\mathbf{m}}(s) := \sum_{x \in G(F)_{\mathbf{m}}} H_L(x)^{-s}$

- Let $\delta_{\mathbf{m}}$ characteristic function of $G(F)_{\mathbf{m}}$.
- Characters of $G(\mathbb{A}_F)$: $\{\chi_y : y \in G(\mathbb{A}_F)\}$.
- Fourier transform: $\widehat{H}_{\mathbf{m}}(y, s) = \int_{G(\mathbb{A}_F)} H(x)^{-s} \delta_{\mathbf{m}}(x) \chi_y(x) \mathrm{d}x$.

Theorem (Tate)

($*$) If $\sum_{x \in G(F)} H_L(x+z)^{-s} \delta_{\mathbf{m}}(x+z)$ converges absolutely for z in a fundamental domain $\mathbf{K}$ for $G(\mathbb{A}_F)/G(F)$, and

($\widehat{*}$) if $\sum_{y \in G(F)} \widehat{H}_{\mathbf{m}}(y, s)$ converges absolutely,

then $Z_{\mathbf{m}}(s) = \sum_{y \in G(F)} \widehat{H}_{\mathbf{m}}(y, s)$.

For ($*$) we find $\mathbf{K}$ such that $H(\cdot)^{-s}$ and $\delta_{\mathbf{m}}$ are $\mathbf{K}$-periodic + absolute convergence of $Z_{\mathbf{m}}(s)$ for $\Re(s) \gg 1$.

$\rightsquigarrow$ Suffice to study $\sum_{y \in G(F)} \widehat{H}_{\mathbf{m}}(y, s)$ $\begin{cases} \text{absolute convergence for } \Re(s) \gg 1, \\ \text{rightmost pole.} \end{cases}$

Reduction to a discrete sum

If χ_y is nontrivial on $\mathbf{K}$, then $\widehat{H}_{\mathbf{m}}(y,s)=0$, and

$$\Lambda = \{y \in G(F) : \chi_y \text{ trivial on } \mathbf{K}\}$$

is an $\mathcal{O}_F$-module of full rank in $G(F)$.

$\rightsquigarrow$ Study $\sum_{y\in\Lambda} \widehat{H}_{\mathbf{m}}(y,s) = \widehat{H}_{\mathbf{m}}(0,s) + \sum_{y\in\Lambda\smallsetminus\{0\}} \widehat{H}_{\mathbf{m}}(y,s)$.

Proof strategy:

Poisson summation

$$Z_{\mathbf{m}}(s) = \widehat{H}_{\mathbf{m}}(0,s) + \sum_{y\in\Lambda\smallsetminus\{0\}} \widehat{H}_{\mathbf{m}}(y,s)$$

pole of order b at $s=a$

absolute convergence,
pole of order $>b$ at $s=a$

Study of $\sum_{y\in\Lambda} \widehat{H}_{\mathbf{m}}(y,s)$ $\begin{cases}\text{absolute convergence for } \Re(s) \gg 1,\\ \text{rightmost pole.}\end{cases}$

Recall: $\widehat{H}_{\mathbf{m}}(y,s) = \int_{G(\mathbb{A}_F)} H(x)^{-s}\delta_{\mathbf{m}}(x)\chi_y(x)\mathrm{d}x = \prod_v \widehat{H}_{\mathbf{m},v}(y,s)$.

Recall: $a = \max_{\alpha\in\mathcal{A}} \frac{\rho_\alpha - (1-\frac{1}{m_\alpha})}{\lambda_\alpha}$, $b = \#\left\{\alpha \in \mathcal{A} : \frac{\rho_\alpha - (1-\frac{1}{m_\alpha})}{\lambda_\alpha} = a\right\}$.

Case $y = 0$ ($\rightsquigarrow$ rightmost pole)

$\prod_{v\notin S} \widehat{H}_{\mathbf{m},v}(0,s) = \prod_{\alpha\in\mathcal{A}} \zeta_{F_\alpha}(m_\alpha(s\lambda_\alpha - \rho_\alpha + 1)) \cdot \left(\begin{matrix}\text{holomorphic}\\ \text{for } \Re(s) > a-\varepsilon\end{matrix}\right)$.

If $v \in S$, $\widehat{H}_{\mathbf{m},v}(0,s)$ holomorphic for $\Re(s) > a - \varepsilon$, as $m_\alpha < +\infty$ $\forall \alpha \in \mathcal{A}$.

Case $y \in \Lambda \smallsetminus \{0\}$ ($\rightsquigarrow$ absolute convergence)

$\widehat{H}_{\mathbf{m}}(y,s) = F(s)\cdot R(y,s)$ with

$F(s)$ holomorphic for $\Re(s) \geq a$ except for a pole of order $> b$ at $s = a$,

$|R(y,s)| \ll \frac{(1+|s|)^{M_N}}{(1+H_{L,\infty}(y))^N}$ for $\Re(s) > a-\varepsilon$ and $N \gg 1$.