Bordeaux seminar talk 13/11/2020

"Campana points, a new number theoretic challenge"

Setting

k number field

S finite set of primes

$R = \mathcal{O}_{k,S}$

$$\begin{array}{ccccccl} & & \text{reduced snc divisor} & & & & \\ \sum_{i=1}^{s} D_i & = & D & \subset & X/_k & & \text{smooth proj} \\ \downarrow & & \downarrow & & \downarrow & & \\ \sum_{i=1}^{s} \mathcal{D}_i & = & \mathcal{D} & \subset & \mathcal{X}/R & & \text{regular proper flat} \end{array}$$

closure of D_i in $\mathcal{X}$

§ Integral points vs rational points

Rmk: $\mathcal{X}/R$ proper $\Rightarrow$ $\mathcal{X}(R) = \mathcal{X}(k)$

but $(\mathcal{X} \setminus \mathcal{D})(R) \subsetneq (\mathcal{X} \setminus \mathcal{D})(k)$

Eg: $k = \mathbb{Q}$, $R = \mathbb{Z}$ $\quad \mathcal{X} = \mathbb{P}^1 = \mathbb{A}^1 \cup \underbrace{\{\infty\}}_{\mathcal{D}}$, $\infty = (0:1)$

$$(\mathcal{X} \setminus \mathcal{D})(\mathbb{Z}) = \mathbb{A}^1(\mathbb{Z}) = \mathbb{Z} \subsetneq \mathbb{Q} = \mathbb{A}^1(\mathbb{Q}) = (\mathcal{X} \setminus \mathcal{D})(\mathbb{Q})$$

A point $x \in \mathcal{X}(R)$ is integral w.r.t. $\mathcal{D}$ if $x \notin \mathcal{D}$ modulo every prime $\notin S$

Eg: $(2:1) \underset{\text{mod } 2}{\rightsquigarrow} (0:1)$ not integral.

§ Campana points.

Slogan:
- Campana points interpolate between integral and rational points
- Campana points are rational points with special "tangency conditions" w.r.t. $\mathcal{D}$

Recall $\mathcal{X}$, $\mathcal{D} = \sum_{i=1}^{s} \mathcal{D}_i$, $\underline{m} = (m_1, \dots, m_s)$ $\quad m_i \in \mathbb{Z}_{>0}$

<u>Def</u>: A point $x \in \mathcal{X}(R)$ is Campana w.r.t. $(\mathcal{D}, \underline{m})$ if $\forall\, i = 1, \dots, s$ and $\forall$ prime $p \notin S$

$$x \in \mathcal{D}_i \bmod p \implies x \in \mathcal{D}_i \bmod p^{m_i}.$$

Eg: $\mathcal{X} = \mathbb{P}^1$ $\quad \mathcal{D} = \{\infty\}$ $\quad k = \mathbb{Q}$, $R = \mathbb{Z}$

$$\mathbb{A}^1(\mathbb{Q}) = \mathbb{Q} = \left\{ \frac{a}{b} : a, b \in \mathbb{Z} \text{ coprime}, b \neq 0 \right\}$$

$(b:a) \in \mathbb{P}^1$

$$\{p \mid b \Rightarrow p^2 \mid b\} \qquad m = 2$$
$$\cup\!\!\!\mid$$
$$\{p \mid b \Rightarrow p^3 \mid b\} \qquad m = 3$$
$$\vdots$$
$$\cup\!\!\!\mid$$
$$\bigcap_{m \geq 1} \underbrace{\{p \mid b \Rightarrow p^m \mid b\}}_{b \text{ is } \underline{m\text{-full}}} = \{b = \pm 1\} = \mathbb{Z} = \mathbb{A}^1(\mathbb{Z})$$

Eg: $\mathcal{X} = \mathbb{P}^1$, $\quad \mathcal{D} = \{0, 1, \infty\}$
$\quad \underline{m} = \{m_1, m_2, m_3\}$

(*) $\{(b:a) : a, b \in \mathbb{Z}$ coprime, a m_1-full, $a-b$ m_2-full, b m_3-full$\}$

<u>Question</u>: If X is a curve and $X(k)$ is infinite, under what conditions is the set of Campana points finite?

Eg: $(\mathbb{P}^1 \setminus \{0, \infty\})(\mathbb{Z}) = \{(1:1), (1:-1)\}$

<u>Orbifold Mordell conj</u> (Campana) $\quad X$ curve

If $K_X + \sum_{i=1}^{s} \left(1 - \frac{1}{m_i}\right) D_i$ is ample, the set of Campana points is finite.

Rmk: Orbifold Mordell follows from the abc-conj.

Eg: ⊛ for $m_1, m_2, m_3 \geq 4$ conj still open

Thm (Campana 2005 (char 0), Kebekus-Smeets-Pereira 2019 char > 0)

Orbifold Mordell holds over function fields.

Manin-type conj (P.-Smeets-Tanimoto-Várilly-Alvarado 2020)

Assume: $-\left(K_X + \sum_{i=1}^{s}\left(1-\frac{1}{m_i}\right)D_i\right)$ ample

Campana points are dense in X

L adelically metrized line bundle ample

then $\exists$ thin $Z \subseteq X(k)$ s.t.

contained in a finite union of proper subvarieties and images $f(Y(k))$ of $f: Y \to X$ generically finite of degree ≥ 2

$\#\{\text{Campana points}^{*} \notin Z : H_L(x) \leq B\}$

$\sim c\, B^{a} (\log B)^{b}$ — geometric invariants, $c > 0$

$\left(aL + K_X + \sum_{i=1}^{s}\left(1-\frac{1}{m_i}\right)D_i\right)$

H_L induced by $X \xhookrightarrow{|L|} \mathbb{P}^N$ and the Weil height.

Open problem: $\mathcal{X} = \mathbb{P}^1$ $\mathcal{D} = \{0, 1, \infty\}$ $m_1 = m_2 = m_3 = 2$

we expect that $\sim c\, B^{1/2}$

(Browning-Van Valckenborgh 2012): $\gg B^{1/2}$, $\ll B^{3/5} (\log B)^{12}$ $k = \mathbb{Q}$

Higher dimension: $k = \mathbb{Q}$ $\mathcal{X} = \mathbb{P}^{n-2}$, $\mathcal{D}_i = \{x_i = 0\}$ $i = 1, \dots, n-1$

$\mathcal{D}_n = \{\sum_{i=1}^{n-1} x_i = 0\}$

(Van Valckenborgh 2012): conj holds for $n \geq 5$, $m_1 = \dots = m_n = 2$.

(Browning-Yamagishi 2019): arbitrary m_i's + technical.

Manin type conj holds

- smooth compactifications of vector groups with equivariant boundary, k arbitrary #fields (P.- Smeets - Tanimoto - Várilly Alvarado)
- many split toric varieties $/\mathbb{Q}$ with equiv. boundary and log-anticanonical height. (P.- Schindler).
- biequivariant compactifications of the Heisenberg group$/\mathbb{Q}$ with equivariant boundary (Xiao 2020)
- E/k extension of number fields of degree d

$\rightsquigarrow$ asymptotic formula for $\mathcal{X} = \mathbb{P}^{d-1}$ $\mathcal{D} = \{ N_{E/k} = 0 \}$
$m \geq 2$ coprime to d.

(Streeter 2020)

Open problems: Fano type situation

- Do Campana points satisfy potential density?
- If Zariski dense, is the set of Campana points not thin?
- Do Campana points satisfy weak-weak approximation.

<u>Thm</u> (Nakahara - Streeter 2020)

If $(\mathcal{X}, \mathcal{D})$ satisfies weak weak approx. the set of Campana points is not thin whenever it is nonempty.