UCG seminar talk 10/11/2020

"Rational points on Fano varieties"

$X = \{ f_1 = \dots = f_r = 0 \} \subseteq \mathbb{P}^n$ proj alg. variety $/k$

$X(k) = \{ x \in \mathbb{P}^n(k) : f_1(x) = \dots = f_r(x) = 0 \}$ set k-points.

Eg: $X_\pm = \{ x^2 + y^2 \pm z^2 = 0 \} \subseteq \mathbb{P}^2$, $X_+(\mathbb{Q}) = \phi$, $X_-(\mathbb{Q}) \neq \phi$

$\exists\, k/\mathbb{Q}$ finite s.t. $\underset{\substack{\cup \\ (1:i:0)}}{X_+(k)} \neq \phi$ eg. $k = \mathbb{Q}(i) \subseteq \mathbb{C}$

Slogan: geometry determines arithmetic.

Eg: Fermat curves $X_d = \{ x^d + y^d - z^d = 0 \} \subseteq \mathbb{P}^2/\mathbb{Q}$

$d=1$ line

$d=2$ conic $\mathbb{Q}$-points $\leftrightsquigarrow$ Pythagorean triples $\Big\}$ $X_d(\mathbb{Q})$ infinite

$d=3$ elliptic curve

$d \geq 4$ genus $= \dfrac{(d-1)(d-2)}{2} \geq 2$ $\Big\}$ $X_d(\mathbb{Q})$ is finite

Curves:	genus 0	genus 1	genus ≥ 2
for $k/\mathbb{Q}$ finite	many points	many points (less)	fin. many points (Faltings)
	many global "antidiff forms"	1 global diff form	many global diff forms
	$-K_X$ ample	K_X trivial	K_X ample

Higher dimension:

	Fano	Calabi-Yau type	varieties of gen. type
	$-K_X$ ample	K_X (num.) trivial	K_X big (eg. ample)
Eg.	$\mathbb{P}^n$, cubic surf.	abelian varieties, K3 surfaces, quartic surf.	quintic surf.

Fano	Calabi-Yau type	general type

Potential density conj:
$\exists\, k/\mathbb{Q}$ finite s.t.
$X(k)$ is not contained in any proper subvariety
(Zariski density)

Bombieri-Lang conj:
$X(k)$ contained in a proper subvariety $\forall\, k/\mathbb{Q}$ finite

these varieties are the building blocks of the birational classification of proj alg. varieties.

In char 0 MMP + Abundance conj.

$$\begin{tikzcd} U \arrow[r, "\text{isom}"] & U' \\ X \arrow[u, phantom, "\cap"] \arrow[r, dashed, "\text{bir}"] & X' \arrow[u, phantom, "\cap"] \arrow[d, "\pi"] \\ & Y \arrow[r, dashed] & Y' \arrow[d, dashed] \\ & & Z \end{tikzcd}$$

X rat. conn.

$\dim Y < \dim X$ $\qquad$ Y rat. conn.

① X, X' of general type
or ② fibers of π are C-Y type
or ③ " " " Fano

<u>Fano varieties</u> build the class of <u>rationally connected</u> varieties
not bir. inv $\qquad\qquad$ bir. inv.

Relation to rat. points

<u>Rmk</u>: If X satisfies potential density then also X' does.

<u>Thm</u> (Lang-Nishimura): the existence of rat. points is a birational invariant for smooth proj varieties.

Expectation for $k/\mathbb{Q}$ finite:

- X smooth Fano, $X(k) \neq \emptyset \Rightarrow X(k)$ Zariski dense
  ($\Rightarrow$ infinite)
  - .. X CY type: counterexample Fermat curve $X_3(\mathbb{Q}) \neq \emptyset$ finite

- X rationally connected (eg. Fano),
  $X(k)$ Zariski dense $\Rightarrow X(k)$ not thin, i.e.
  not contained in a finite union of proper subvarieties and images $f(Z(k))$ for $f: Z \to X$ generically finite of degree > 1

Eg: $\{(x:y) \in \mathbb{P}^1(\mathbb{Q}) : x, y \text{ squares}\}$ is thin

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ (u:v) &\longmapsto (u^2:v^2) \end{aligned}$$

.. X CY type: X abelian varieties $\Rightarrow X(k)$ thin
X K3 surf $\not\Rightarrow X(k)$ thin

Theorem (Browning–Loughran 2019)
X Fano and rational points are equidistributed on $U \subseteq X$ open
and $Z \subseteq U(k)$ thin, then

$$\lim_{B \to \infty} \frac{\#\{x \in Z : H(x) \leq B\}}{\#\{x \in U(k) : H(x) \leq B\}} = 0 \qquad /k/\mathbb{Q} \text{ finite}$$

Here, $H: X(k) \longrightarrow \mathbb{R}_{\geq 0}$ s.t. $\#\{x \in X(k) : H(x) \leq B\}$ finite $\forall B \in \mathbb{R}_{\geq 0}$

anticanonical height.

<u>Conj</u> (Manin et al. 1989)

X Fano, $X(k)$ Zariski dense, then $\exists\, Z \subseteq X(k)$ thin s.t. $\#\{x \in X(k) \setminus Z : H(x) \leq B\} \sim c\, B(\log B)^{r-1}$

r: rank $\mathrm{Pic}(X)$

$c = \alpha \cdot \beta \cdot \tau$

α: geom, β: cohom, τ: Tamagawa volume

<u>Theorem</u> (Peyre)

Equidistribution of rational points holds X if Manin's conj holds for every anticanonical height H on X.

Known for flag varieties (eg. Grassmannians)

toric varieties (eg. $\mathbb{P}^n$, $\mathbb{P}^1 \times \mathbb{P}^1$, $Bl_x \mathbb{P}^n$)

equivariant compactifications of vector groups

complete intersections of low degree w.r.t. dim.

My work: study of Manin's conj via Cox rings.

<u>Thm</u> (P–Frei (a=1) 2016, P.–Derenthal 2020)

the surface $\{x_0x_4 + x_1^2 - a x_3^2 = x_2x_3 - x_4^2 = 0\} \subseteq \mathbb{P}^4$ satisfy Manin's conj over all $k/\mathbb{Q}$ finite $\forall\, a \in k \setminus \{0\}$.

proof sketch:

$$\mathbb{A}^N \underset{\text{closed}}{\supseteq} \underbrace{\operatorname{Spec} R(X)}_{\text{Cox ring}} \underset{\text{open}}{\supseteq} Y \xrightarrow[\text{for } \mathbb{G}_m^r]{\text{torsor}} X \qquad r = \mathrm{rk}\,\mathrm{Pic}(X)$$