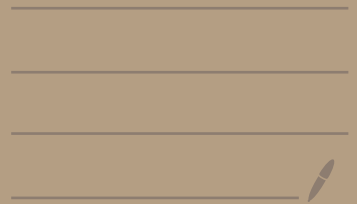


UCG seminar talk 10/11/2020



"Rational points on Fano varieties"

$$X = \{ f_1 = \dots = f_r = 0 \} \subseteq \mathbb{P}^n \quad \text{proj alg. variety / } k$$

$$X(k) = \{ x \in \mathbb{P}^n(k) : f_1(x) = \dots = f_r(x) = 0 \} \quad \text{set } k\text{-points}$$

Eg: $X_{\pm} = \{ x^2 + y^2 \pm z^2 = 0 \} \subseteq \mathbb{P}^2$, $X_+(\mathbb{Q}) = \emptyset$, $X_-(\mathbb{Q}) \neq \emptyset$
 $\exists k/\mathbb{Q}$ finite s.t. $X_+(k) \neq \emptyset$ eg. $k = \mathbb{Q}(i) \subseteq \mathbb{C}$
 $(1:i:0)$

Slogan: geometry determines arithmetic.

Eg: Fermat curves $X_d = \{ x^d + y^d - z^d = 0 \} \subseteq \mathbb{P}^2 / \mathbb{Q}$

$d=1$ line

$d=2$ conic \mathbb{Q} -points are Pythagorean triples } $X_d(\mathbb{Q})$ infinite

$d=3$ elliptic curve

$d \geq 4$ genus = $\frac{(d-1)(d-2)}{2} \geq 2$ } $X_d(\mathbb{Q})$ is finite

Curves: genus 0

genus 1

genus ≥ 2

for k/\mathbb{Q} finite many points

many points (less)

fin. many points (Faltings)

many global "anti-diff forms" $-K_X$ ample

1 global diff form K_X trivial

many global diff forms K_X ample

Higher dimension:

Fano

Calabi-Yau type

varieties of gen. type

$-K_X$ ample

K_X (num.) trivial

K_X big (eg. ample)

Eg. \mathbb{P}^n cubic surf.

abelian varieties
 $K3$ surfaces
 quartic surf.

quintic surf.

Expectation for k/\mathbb{Q} finite:

- X smooth Fano, $X(k) \neq \emptyset \Rightarrow X(k)$ Zariski dense
(\Rightarrow infinite)
- X CY type: counterexample Fermat curve $X_3(\mathbb{Q}) \neq \emptyset$
finite
- X rationally connected (eg. Fano),
 $X(k)$ Zariski dense $\Rightarrow X(k)$ not thin, i.e.
not contained in a finite union
of proper subvarieties and images
 $f(Z(k))$ for $f: Z \rightarrow X$
generically finite of degree > 1

Eg: $\{(x:y) \in \mathbb{P}^1(\mathbb{Q}) : x, y \text{ squares}\}$ is thin

$$\begin{array}{ccc} \mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 \\ (u:v) & \longmapsto & (u^2:v^2) \end{array}$$

- X CY type: $\begin{cases} X \text{ abelian varieties} \Rightarrow X(k) \text{ thin} \\ X \text{ K3 surf} \not\Rightarrow X(k) \text{ thin} \end{cases}$

Theorem (Browning-Loughran 2019)

X Fano and rational points are equidistributed on $U \subseteq X$
open

and $Z \subseteq U(k)$ thin, then

$$\lim_{B \rightarrow \infty} \frac{\#\{x \in Z : H(x) \leq B\}}{\#\{x \in U(k) : H(x) \leq B\}} = 0 \quad / k/\mathbb{Q} \text{ finite}$$

Here, $H: X(k) \rightarrow \mathbb{R}_{\geq 0}$ s.t. $\#\{x \in X(k) : H(x) \leq B\}$
finite $\forall B \in \mathbb{R}_{\geq 0}$
↑
anticanonical height.

