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"The split torsor method for Manin's conjecture"

- the Weil height

Recall : $\#\{(x_0,\dots,x_n)\in\mathbb{Z}^{n+1} : |x_0|,\dots,|x_n|\le B\}$ finite $\forall B>0$

$$H : \mathbb{P}^n(\mathbb{Q}) \longrightarrow \mathbb{R}_{>0}$$

$$(x_0:\dots:x_n) \longmapsto \prod_{\substack{v \text{ place}\\ \text{of } \mathbb{Q}}} \max_{0\le i\le n} |x_i|_v = \max_{0\le i\le n}(|x_0|,\dots,|x_n|)$$

if $x_0,\dots,x_n\in\mathbb{Z}$ and $\gcd(x_0,\dots,x_n)=1$

Similarly, if k number field

$$H(x_0:\dots:x_n) = \prod_{\substack{v \text{ place}\\ \text{of } k}} \max_{0\le i\le n} |x_i|_v \qquad \forall\,(x_0:\dots:x_n)\in\mathbb{P}^n(k).$$

- Heights are defined by line bundles

if $X \overset{\varphi}{\subseteq} \mathbb{P}^n$ $\rightsquigarrow$ $\#\{x\in X(k) : \overset{H(\varphi(x))}{H(x)} \le B\}$

asymptotic behavior for $B\to\infty$?

height on X depends on the embedding φ, which is defined by $\varphi^*\mathcal{O}(1)=\mathcal{L}$ ample line bundle on X and its global sections $x_0,\dots,x_n$.

$\rightsquigarrow$ "ample" line bundles define heights.

- All varieties are equipped with a canonical line bundle and an anticanonical line bundle, its dual.

X Fano if the anticanonical line bundle is ample

• <u>Conjecture</u> (Manin-Batyrev-Peyre-Tschinkel, Chambert-Loir, Le Rudulier, Tanimoto, Lehmann)

X smooth Fano-type variety / number field k

$\mathcal{L}$ "ample" line bundle on X

If $X(k)$ Zariski dense in X and not thin, then $\exists\, Z \subset X(k)$ thin s.t.

$$\#\{x \in X(k) \setminus Z : H_{\mathcal{L}}(x) \leq B\} \underset{B\to\infty}{\sim} c_{X,H}\, B^{a} (\log B)^{b}$$

$\neq 0$

where a, b given geometric invariants of X.

Thin set: Z thin if contained in a finite union of subvarieties of X and of images $f(Y(k))$ for $f: Y \to X$ generically finite maps of degree >1

Eg: $\{(x:y) \in \mathbb{P}^1(\mathbb{Q}) : x, y \text{ squares}\}$ is thin

$$\begin{aligned} \mathbb{P}^1 &\longrightarrow \mathbb{P}^1 \\ (u:v) &\longmapsto (u^2:v^2) \end{aligned}$$

• <u>History</u>: Birch '62 hypersurfaces in $\mathbb{P}^n_{\mathbb{Q}}$

Schanuel '79 $\mathbb{P}^n_k$

conjecture ⇝ Franke-Manin-Tschinkel '89 flag varieties

$\mathcal{L}$ = antican.

Z closed

• Batyrev-Manin '90 : general $\mathcal{L}$

• Peyre '95 : constant

• Batyrev-Tschinkel '96 counterexample

Astérisque 251 '98

Salberger : toric varieties via torsors

Peyre 2001 : universal torsors and circle method

Derenthal 2006 : universal torsors via Cox rings de la Bretèche - Browning - Peyre 2012 : Châtelet surfaces

- Le Rudulier 2014 $\mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ Z thin
- ★ Derenthal - Frei 2014 : universal torsor method / $\mathbb{Q}(i)$
- ★ Frei - P. 2016 : universal torsor method / k
- Lehmann - Tanimoto 2017 : Z thin
- Derenthal - P. 2019 : split torsors via Cox rings
- ★ Derenthal - P. 2020 : split torsor method / k

- Cox rings

Cox ring = ring of global sections of all line bundles of X

Eg: $\mathbb{P}^1 \times \mathbb{P}^1$ $(x_0:x_1)$ $(y_0:y_1)$ line bundles $\mathcal{O}(a,b)$ $(a,b) \in \mathbb{Z}^2$

global sections = bihomog. polynomials of degree (a,b)

(eg: $x_0^2 y_0 - x_0 x_1 y_1$ for $\mathcal{O}(2,1)$)

Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ $= k[x_0, x_1, y_0, y_1]$

Why ? ★ if I know all global sections of all line bundles I can describe all the heights

★ if Cox ring is finitely generated (it is for Fanos) then I can describe the split torsor of X.

- The split torsor method :

X Fano $\Rightarrow$ $\mathrm{Pic}(X) \cong \mathbb{Z}^r$ fin. gen.

Cox ring $R(X)$ fin. gen /k

$R(X) = k[z_1, \dots, z_N] / (g_1, \dots, g_s)$

$$\underset{(z_1,\dots,z_N)}{\mathbb{A}^N_k} \supseteq \{g_1 = \dots = g_s = 0\} \setminus \{f_1 = \dots = f_t = 0\} = Y \xrightarrow{\mathbb{G}_m^r\text{-torsor}} X \quad /k$$

$$\begin{tikzcd} Y \arrow[r] \arrow[d, dash] & X \arrow[d, dash] & /k \\ \mathcal{Y} \arrow[r] & \mathcal{X} & /\mathcal{O}_k \end{tikzcd}$$

$$\mathbb{A}^N_{\mathcal{O}_k} \supseteq \mathcal{Y}$$

$$\underset{\text{lattice}}{\mathcal{O}_k^N} \supseteq \underset{\text{explicit description}}{\mathcal{Y}(\mathcal{O}_k)}$$

$Y(k) \overset{\pi}{\twoheadrightarrow} X(k)$, but $\mathcal{Y}(\mathcal{O}_k) \xrightarrow{\pi} \mathcal{X}(\mathcal{O}_k)$ not surjective in general

$$\underset{\substack{c \in Cl(\mathcal{O}_k)^r \\ \text{twists}}}{\bigsqcup} \mathcal{Y}^c(\mathcal{O}_k) \overset{\sqcup_c \pi^c}{\twoheadrightarrow} \mathcal{X}(\mathcal{O}_k)$$

$\rightsquigarrow$ bijections

$$Y(k)/\mathbb{G}_m^r(k) = X(k)$$

$$\bigsqcup_c \mathcal{Y}^c(\mathcal{O}_k)/\underbrace{\mathbb{G}_m^r(\mathcal{O}_k)}_{(\mathcal{O}_k^\times)^r} = \mathcal{X}(k) \quad \text{if } \mathcal{X} \text{ proper}/\mathcal{O}_k$$

- if $\mathcal{O}_k^\times$ finite: $\#\{x \in X(k) \setminus Z : H(x) \le B\} =$
$$= \frac{1}{(\#\mathcal{O}_k^\times)^r} \sum_c \#\{y \in \mathcal{Y}^c(\mathcal{O}_k) \setminus \pi^{c\,-1}(Z) : H(y) \le B\}$$

- if $\mathcal{O}_k^\times$ infinite: need to work with a fundamental domain for the action of $(\mathcal{O}_k^\times)^r$