# Lecture notes on Cosmology (ns-tp430m)

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## Part IV: Cosmological Perturbations

In this chapter we shall show how to calculate the spectrum of cosmological perturbations generated during a primordial epoch of cosmic inflation. We shall then outline how primordial cosmological perturbations source the observed temperature fluctuations in the CMB as well as how they seed the large scale structure of the Universe.

Cosmological perturbations are created by the amplification of quantum fluctuations of matter and metric perturbations during inflation. Therefore, in order to understand their creation, it is necessary to study the evolution of small perturbations of homogeneous metric and matter fields. Depending on how they transform under spatial rotations and local time shifts, cosmological perturbations can be divided into scalar, vector and tensor perturbations. Here we shall assume that matter is made up of a single real scalar field, which gives rise to the simplest matter perturbation and a scalar metric perturbation. The inflaton perturbation is most likely the most important matter perturbation, since there is evidence that it is precisely the inflaton perturbation that sources both the CMB temperature fluctuations and structure formation.

We begin by quoting the Einstein-Hilbert action for gravity plus scalar matter,

$$S[g_{\mu\nu},\Phi] = -\frac{M_P^2}{2}\int d^4x\sqrt{-g}R(g_{\mu\nu}) + \int d^4x\sqrt{-g}\Big(\frac{1}{2}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) - V(\Phi)\Big)\,, \tag{1}$$

where $R$ denotes the Ricci scalar, $V(\Phi)$ is a scalar potential, and $g_{\mu\nu}$ and $\Phi$ denote metric tensor and scalar field, respectively. For simplicity, in (1) we set the cosmological term $\Lambda = 0$.

### A. The scalar, vector, tensor decomposition and gauge invariance

The theory of cosmological perturbations is based on the assumption that the decomposition into the background and fluctuating fields,

$$g_{\mu\nu}(x) = g^b_{\mu\nu}(t) + \delta g_{\mu\nu}(x)\,;\qquad \Phi(x) = \phi(t) + \varphi(x)\,, \tag{2}$$

is justified. This is the case when the components of $\delta g_{\mu\nu}$ and $\varphi$ are small, in the sense that $\delta g_{\mu\nu} \ll 1$ and $\varphi \ll \phi$. Assuming that perturbations are generated by quantum fluctuations, during inflation this is indeed the case. In quantum field theory this method is known as the background field method.

When the background metric is taken to correspond to a homogeneous flat cosmology, which in cosmological (comoving) time ($t$) and conformal time ($\tau$) reads,

$$g^b_{\mu\nu}(t) = \mathrm{diag}(1, -a^2(t), -a^2(t), -a^2(t))\,; \qquad g^b_{\mu\nu}(\tau) = a^2(\tau)\eta_{\mu\nu}\,, \tag{3}$$

the background equations of motion can be obtained by varying the action (1). The result is well known,

$$\begin{aligned} H^2(t) &= \frac{\rho_b}{3M_P^2}\,, \quad \rho_b = \frac{1}{2}\dot\phi^2 + V(\phi) \\ \dot H &= -\frac{\dot\phi^2}{2M_P^2}\,, \quad \ddot\phi + 3H\dot\phi + V'(\phi) = 0 \end{aligned} \tag{4}$$

As discussed in part III, these equations can be solved for the inflaton $\phi$ in slow roll approximation, and provided the initial value of the inflaton is large enough, $|\phi| \gg M_P$, and the potential flat enough, one will get a period of primordial inflation. It is well known that when the action (1) is expanded in powers of fluctuations, the structure of the first other action is of the form: the equations of motion multiplied by the linear perturbations, and thus present no new information. On the other hand, the action for second order perturbations provides an essential information on the dynamics of fluctuations during inflation. The amplitude of fluctuations can be normalised by the procedure of canonical quantisation. But before we show how to do that, we pause to discuss diffeomorophism invariance of fluctuations.

### *1. The Sasaki-Mukhanov field*

We shall now study how different metric and scalar perturbations transform under the infinitesimal coordinate transformations, also known as (gravitational) gauge transformations. To that purpose let us consider an infinitesimal coordinate transformation (coordinate shift),

$$x^\mu \to \tilde x^\mu = x^\mu + \xi^\mu(x)\,, \tag{5}$$

where $\xi^\mu = \xi^\mu(x)$ is an arbitrary (infinitesimal) vector function. Under these transformations the field and the metric tensor transform as,

$$\tilde\Phi(\tilde x) = \Phi(x)\,; \qquad \tilde g^{\mu\nu}(\tilde x) = g^{\rho\sigma}(x)\Big[\delta^\mu_\rho + \frac{\partial\xi^\mu(x)}{\partial x^\rho}\Big]\Big[\delta^\nu_\sigma + \frac{\partial\xi^\nu(x)}{\partial x^\sigma}\Big]\,. \tag{6}$$

Making use of $\tilde\Phi(\tilde x) = \tilde\Phi(x) + \xi^\rho\partial_\rho\tilde\Phi(x) + \mathcal{O}(\xi^2)$ and the analogous relation for the metric tensor, and neglecting terms that are quadratic or higher order in $\xi$, Eq. (6) can be recast as,

$$\tilde\Phi(x) = \Phi(x) - \xi^\rho\partial_\rho\Phi(x)\,; \qquad \tilde g^{\mu\nu}(x) = g^{\mu\nu}(x) + g^{\rho\nu}(x)\frac{\partial\xi^\mu(x)}{\partial x^\rho} + g^{\mu\rho}(x)\frac{\partial\xi^\nu(x)}{\partial x^\rho} - \xi^\rho(x)\frac{\partial g^{\mu\nu}(x)}{\partial x^\rho}\,. \tag{7}$$

Making use of the connection,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\Big[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\Big] \tag{8}$$

the second equation in (7) can be rewritten as,

$$\tilde{g}^{\mu\nu}(x) = g^{\mu\nu}(x) + \nabla^\mu \xi^\nu(x) + \nabla^\nu \xi^\mu(x)\,, \tag{9}$$

or

$$\tilde{g}_{\mu\nu}(x) = g_{\mu\nu}(x) - \nabla_\mu \xi_\nu(x) - \nabla_\nu \xi_\mu(x)\,. \tag{10}$$

where we used $\tilde{g}_{\mu\nu}\tilde{g}^{\nu\rho} = \delta_\mu^{\ \rho}$.

Let us now look in detail how the field fluctuations $\varphi$ and $\delta g_{\mu\nu}$ in (2) (which can be of the same order as $\xi^\nu$) transform under the coordinate transformation (5). For the scalar field Eq. (7) holds, such that scalar field fluctuations transform as,

$$\varphi(x) \to \tilde{\varphi}(x) = \varphi(x) - \phi'(\tau)\xi^0 = \varphi(x) - \dot{\phi}(t)\frac{\xi_0}{a}\,, \tag{11}$$

where $\phi' = d\phi/d\tau$ and $\dot{\phi} = d\phi/dt = \phi'/a$ and $\xi^0 = \xi_0/a^2$. Here we will be primarily interested in the spatial part of the metric tensor perturbation, $\delta g_{ij} = a^2 h_{ij}$ which, according to (10), transforms as

$$a^2 h_{ij} \to \widetilde{a^2 h_{ij}} = a^2 h_{ij} - \nabla_i \xi_j - \nabla_j \xi_i = a^2 h_{ij} - \partial_i \xi_j - \partial_j \xi_i + 2\frac{a'}{a}\delta_{ij}\xi_0\,, \tag{12}$$

where to get the last equality we used, $\Gamma^0_{ij} = (a'/a)\delta_{ij}$ and $\Gamma^l_{ij} = 0$. Now we introduce the following scalar-vector-tensor decomposition of the spatial components of the metric tensor,

$$h_{ij} = 2\psi\delta_{ij} + 2\partial_i\partial_j E + (\partial_i F_j + \partial_j F_i) + h^{TT}_{ij}\,, \tag{13}$$

where under spatial rotations $\psi$ and $E$ transform as scalars, $F_i$ transforms as a transverse vector ($\partial_i F_i = 0$) and $h^{TT}_{ij}$ is a transverse and traceless tensor:

$$h^{TT}_{ii} = 0\,; \qquad \partial_i h^{TT}_{ij} = 0 = \partial_j h^{TT}_{ij}\,. \tag{14}$$

Upon inserting the decomposition (13) into (12) and breaking the shift vector into the transverse and longitudinal parts,

$$\xi_i = \xi^T_i + \partial_i \xi\,, \qquad \partial_i \xi^T_i = 0\,, \tag{15}$$

we see that the different components of $h_{ij}$ in (13) transform as,

$$F_j \to \tilde{F}_j = F_j - \frac{\xi^T_j}{a^2}\,; \quad E \to \tilde{E} = E - \frac{\xi}{a^2}\,; \quad \psi \to \tilde{\psi} = \psi + \frac{a'}{a^3}\xi_0 = \psi + H\frac{\xi_0}{a}\,; \quad h^{TT}_{ij} \to \tilde{h}^{TT}_{ij} = h^{TT}_{ij}\,. \tag{16}$$

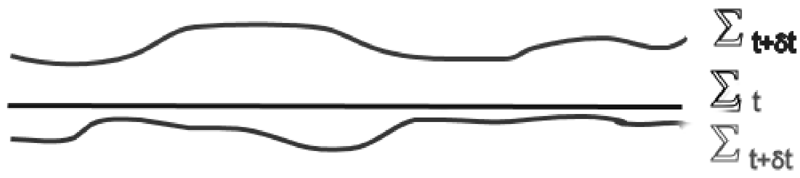


FIG. 1: Time slices.

We have thus learned that, while the scalar and vector spatial metric components transform (and thus can be, at least in principle, set to zero by a suitable coordinante transformation), the tensor components are gauge invariant, and thus can be assigned a physical meaning. Since the conditions (14) represent four conditions on six spatial metric components, $h_{ij}^{TT}$ has two independent components, which represent the two polarisations of the graviton, known as the *plus* $(+)$ and *cross* $(\times)$ polarisations.

Let us now come back to the scalar components. By inspecting eqations (11) and (16) it is easy to see that the following scalar field combination,

$$w_\varphi \equiv \varphi + \frac{\dot{\phi}}{H}\psi \,, \tag{17}$$

or equivalently,

$$w_\psi \equiv \frac{H}{\dot{\phi}} w_\varphi = \psi + \frac{H}{\dot{\phi}}\varphi \,, \tag{18}$$

is gauge invariant. This field is known as the Sasaki-Mukhanov field (or variable), and it plays an important role in the theory of cosmological perturbations [7]. Since $\xi^\mu$ is an arbitrary vector, it can be chosen such to remove some (but not all) of the metric and field components. Since there are four components in $\xi^\mu$, it can be chosen such to remove $F_i$, $E$ and one of the two remaining scalars, *i.e.* $\varphi$ or $\psi$. This is known as gauge fixing. Next we shall discuss two gauge fixing procedures commonly used in literature.

### 2. *The comoving and zero curvature gauge*

From Eq. (12) we see that the gauge function,

$$\xi_j^T = a^2 F_j \,; \qquad \xi = a^2 E \,; \qquad \xi_0 = -\frac{a}{H}\psi \tag{19}$$

fixes a gauge known as the *zero-curvature gauge*, in which $\tilde{F}_j = \tilde{E} = \tilde{\psi} = 0$, and in which the physical dynamical fields are the inflaton $\varphi$ and the graviton $h_{ij}^{TT}$.

On the other hand, chosing the gauge function as,

$$\xi_j^T = a^2 F_j\,; \qquad \xi = a^2 E\,; \qquad \xi_0 = \frac{a}{\dot\phi}\varphi \tag{20}$$

fixes the *comoving gauge*, in which $\tilde F_j = \tilde E = \tilde\varphi = 0$, such that the physical dynamical fields are the spatial gravitational potential $\psi$ and the graviton $h_{ij}^{TT}$. In this gauge the surfaces of constant time are chosen such that an observer does not see any perturbations in the scalar field, *i.e.* the observer is 'comoving' with scalar matter.

In order to get a deeper insight into the physical meaning of the two gauges, we shall first rewrite Eqs. (11) and (16)

$$\varphi(x) \to \tilde\varphi(x) = \varphi(x) - \dot\phi(t)\delta t\,, \qquad \psi \to \tilde\psi = \psi + H\delta t\,, \tag{21}$$

where we made use of $\delta t = a\delta\tau = a\xi^0 = \xi_0/a$. From these it follows that the zero-curvature and comoving gauges correspond to the following choice of time,

$$\begin{aligned} [\delta t(x)]_{\rm zero-curv} &= -\frac{\psi(x)}{H(t)} \qquad ({\rm zero-curvature\;\; gauge}) \\ [\delta t(x)]_{\rm comoving} &= \frac{\varphi(x)}{\dot\phi(t)} \qquad ({\rm comoving\;\; gauge})\,. \end{aligned} \tag{22}$$

In fact, the two gauge choices correspond to two different choices of time. According to coordinate invariance of general relativity, one is free to choose time locally, such that $\delta t(x)$ can be chosen independently on any point is space-time. This freedom is illustrated in figure 1, where we sketch how the space-time manifold $\mathcal{M} = \mathbb{R}\times\Sigma$ is broken into a time direction – corresponding to real numbers $\mathbb{R}$ – and a spatial part – correponding to a three dimensional Riemannian space $\Sigma$. Different gauges then correspond to different choices of constant time hypersurface $\Sigma$, as illustrated in figure 1 [8]. For the two particular choices of time (22), the (gauge invariant) Sasaki-Mukhanov field (18) becomes,

$$w_\psi = \frac{H}{\dot\phi}\varphi \qquad ({\rm zero-curvature\;\; gauge}) \tag{23}$$

$$w_\psi = \psi \qquad ({\rm comoving\;\; gauge}) \tag{24}$$

When calculating cosmological perturbations, one can either work in the gauge invariant formalism, in which case one studies the dynamics of the gauge invariant field (18) or (17). Alternatively, one can fix a gauge according to (22), and then infer the amplitude of the Sasaki-Mukhanov field from (23–24). We shall first show how one performs the latter (gauge fixing) procedure.

But before we do that, we note that, to linear accuracy, the (local) curvature perturbation $w_\psi$ can be also written in terms in terms of $\bar w_\psi$, defined as,

$$\mathrm{e}^{-\bar w_\psi} = 1 - \bar w_\psi + \frac{1}{2}\bar w_\psi \equiv 1 - w_\psi\,. \tag{25}$$

Note that to linear order $\bar{w}_\psi$ and $w_\psi$ are identical, *i,.e.* $\bar{w}_\psi = w_\psi + \mathcal{O}(w_\psi^2)$. From the point of view of metric perturbations, in the so-called Newtonian gauge, in which the only scalar perturbations are $g_{00} = 1 + 2\phi$ and $\psi$, degined by

$$g_{ij}(x) = -a^2(1-2\psi)\delta_{ij} = -\mathrm{e}^{-2(N(t)+\bar{\psi}(x))}\delta_{ij} \, . \tag{26}$$

Here we neglected the graviton $h_{ij}^{TT}$ and gauge fixed the second scalar $E$, the vector $F_i$ to zero. The function

$$N(t) = \int_t^{t_e} dt' H(t') = \ln\left(\frac{a_e}{a}\right) \tag{27}$$

in (26) denotes the usual number of e-folds, and $t_e$ is the time at the end for inflation. From Eq. (26) one would be tempted to associate $\bar{\psi}$ to a local deviation in the number of e-folds from some average (homogeneous) value $N(t)$ induced by the spatial metric perturbation. This generalisation makes sense in the *separate universes* approximation of inflation, according to which, once modes become super-Hubble, they decouple from each other. For each Hubble volume one can define an average expansion rate and follow its evolution on super-Hubble scales as if no other Hubble volume is present. One can argue that this is a reasonable approximation by noting that in the equation of motion for scalar field perturbations, $(\partial_t^2 + 3H\partial_t)\varphi - (\nabla^2/a^2)\varphi \simeq 0$ one can neglect the last (gradient) term on super-Hubble scales since it scales away exponentially fast with time. When one does that, one gets an approximate equation on super-Hubble scales, $(\partial_t^2 + 3H\partial_t)\varphi \simeq 0$ ($\|\nabla\| \ll aH$) which describes the evolution of modes in the separate universes approximation. One can show that an analogous argument holds for the equation of motion for the gauge invariant variable $w_\psi$.

It then follows that, in the separate universes approximation and in the zero-curvature gauge (22), we can make the identification,

$$\bar{w}_\psi(x) = \delta N(x) = H\delta t(x) = H\frac{\delta t}{\delta\phi}\varphi(x) \simeq \frac{H}{\dot{\phi}}\varphi(x) \, . \tag{28}$$

Note that the space-time dependence in (28) is generated by the mapping $\delta t(x)$ between the comoving hypersurface (on which $\delta\phi = 0$) and the zero curvature hypersurface (on which $\psi = 0$). Note further that, up to higher order (quadratic, cubic, *etc*) corrections we have just rederived Eq. (23) (which holds at linear order in perturbations). From the definition (25), Eq. (28) follows immediately when the zero curvature gauge is fixed. In literature one often calculates the spectrum associated with $\bar{w}_\psi$ (rather then with $w_\psi$), because in one field inflationary models $\bar{w}_\psi$ is conserved to all orders on super-Hubble scales during inflation and subsequent radiation and matter eras, which makes the calculations easier. However, the difference between $\bar{w}_\psi$ and $w_\psi$ will be irrelevant in these lecture notes, since we shall calculate the spectrum which involves only the leading order (linear) fields.

### B. Scalar cosmological perturbations in the zero-curvature gauge

We shall now calculate the spectrum of scalar perturbations in the zero-curvature gauge, in which $\tilde{\psi} = 0$ and $w_\psi = (H/\dot{\phi})\varphi$, Eq. (23). The quadratic action for the scalar field perturbations in this gauge can be easily inferred from Eq. (1),

$$S_2[\varphi] = \int d^3x d\tau \Big( \frac{1}{2} a^2 \eta^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) - \frac{1}{2} a^4 V''(\phi)\varphi^2 + \frac{a^2}{2} [-\phi'\varphi' - a^2 V'(\phi)\varphi] h_{00} - a^2 \phi' \varphi (\partial_i h_{0i}) \Big) , \quad (29)$$

where $V'' = d^2V(\phi)/d\phi^2$, $V' = dV(\phi)/d\phi$, and we made use of

$$\sqrt{-g} = a^4 \Big( 1 + \frac{1}{2} h_{00} + \mathcal{O}(h_{\mu\nu}^2) \Big) ; \qquad g^{\mu\nu} = a^{-2} (\eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}((h^{\mu\nu})^2)) , \quad h^{\mu\nu} = \eta^{\mu\rho}\eta^{\rho\sigma} h_{\rho\sigma} , \quad (30)$$

In Eq. (29) we neglected the quadratic terms that do not contain $\varphi$, since we are here not interested in their evolution. Note that, since we have already completely fixed the gauge, we cannot get rid of $h_{00}$ and $h_{0i}$ in (29). This presents a problem in the analysis of the dynamics of $\varphi$. Not taking account of $h_{00}$ will in general lead to wrong results [9]. Nevertheless, there is a (sneaky) way out the impasse, and that is to first perform the analysis in de Sitter space, in which case the problematic terms (containing $h_{00}$ and $h_{0i}$) drop out. This is indeed what we do first below. A separate subsection that follows is devoted to a rigorous analysis, which leads to the same result.

Varying the action (29) gives the equation of motion,

$$(\partial^2 + 2\mathcal{H}\partial_0 + a^2 V'')\varphi = \frac{1}{2} [-\phi'' - 2\mathcal{H}\phi' + a^2 V'(\phi)] h_{00} + \phi' \partial_i h_{0i} , \quad (31)$$

where

$$\partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_0^2 - \nabla^2 , \quad \mathcal{H} = \frac{a'}{a} , \quad \nabla^2 = \partial_i^2 . \quad (32)$$

The term on the right hand side can be simplified by making use of the equation of motion for $\phi$ (4), $\phi'' + 2\mathcal{H}\phi' + V'(\phi) = 0$, resulting in the right hand side,

$$\text{r.h.s. of (31)} = a^2 V'(\phi) h_{00} + \phi' \partial_i h_{0i} . \quad (33)$$

We are here interested in studying the quantum fluctuations of $\varphi$. For that we need the canonical momentum $\pi_\varphi$ of $\varphi$,

$$\pi_\varphi = \frac{\delta S_2}{\delta \varphi'} = a^2 \varphi' . \quad (34)$$

According to the canonical quantisation procedure ($\hbar = 1$),

$$[\hat{\varphi}(\vec{x}, \tau), \hat{\pi}_\varphi(\vec{x}\,', \tau)] = i\delta^3(\vec{x} - \vec{x}\,') . \quad (35)$$

In order determine the amplitude $\varphi$, we need to promote $\varphi$ to a quantum field, solve (31) and impose (35). But in order to do that, we also need to solve for $h_{00}$ and $h_{0i}$. Since we do not have a further information on $h_{00}$ and $h_{0i}$, we shall neither quantise it not solve for it. Instead, we shall study the limit of de Sitter space in which $\varphi$ decouples from $h_{00}$ and $h_{0i}$, and thus (for now) solve our problem. This problem is rigorously dealt with in section C.

We note first that it is more convenient to work with the rescaled field $a\varphi$, which obeys

$$\Big(\partial^2 - \frac{a''}{a} + a^2 V''\Big)(a\hat{\varphi}(x)) = a^2 V'(\phi)(ah_{00}) + \phi' \partial_i (ah_{0i})\,. \tag{36}$$

Secondly, we can use homogeneity of the background space, and expand the field $\varphi$ into Fourier components as,

$$\hat{\varphi}(x) = \int \frac{d^3k}{(2\pi)^3}\Big[\mathrm{e}^{i\vec{k}\cdot\vec{x}}\varphi(k,\tau)\hat{a}_{\vec{k}} + \mathrm{e}^{-i\vec{k}\cdot\vec{x}}\varphi^*(k,\tau)\hat{a}^+_{\vec{k}}\Big]\,, \tag{37}$$

where $\hat{a}_{\vec{k}}$ and $\hat{a}^+_{\vec{k}}$ denote the annihilation and creation operators. While $\hat{a}_{\vec{k}}$ destroys a quantum of $\varphi$ with momentum $\vec{k}$, $\hat{a}^+_{\vec{k}}$ creates a quantum of $\varphi$ with momentum $\vec{k}$. This means that in a vacuum state, $|\Omega\rangle$, which contains no $\varphi$ quanta,

$$\hat{a}_{\vec{k}}|\Omega\rangle = 0\,, \tag{38}$$

*i.e.* $\hat{a}_{\vec{k}}$ destroys the vacuum state of the theory. These operators are the quantum field theory analog of the raising and lowering operators of the simple harmonic oscillator. And similarly to the raising and lowering operators, they obey the simple commutation relations,

$$[\hat{a}_{\vec{k}}, \hat{a}^+_{\vec{k}'}] = (2\pi)^3\delta^3(\vec{k}-\vec{k}'\,)\,, \qquad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}^+_{\vec{k}}, \hat{a}^+_{\vec{k}'}]\,. \tag{39}$$

Upon inserting Eq. (37) into Eq. (36) we obtain,

$$\Big(\frac{d^2}{d\tau^2} + k^2 - \frac{a''}{a} + a^2 V''(\phi)\Big)(a\varphi(k,\tau)) = a^2 V'(\phi)(ah_{00}(k,\tau)) + \phi' \imath k_i (ah_{0i}(k,\tau))\,. \tag{40}$$

Note that, as a consequence of the spatial isotropy of the underlying space-time, the mode functions $\varphi(k,\tau)$ do not depend on the direction of $\vec{k}$, but only on its magnitude. The annihilation and creation operators of course do depend on $\vec{k}$. That also implies that the spectrum, which is what we discuss next, will be a function of the magnitude of momentum, $k = \|\vec{k}\|$, but not of its direction $\vec{k}/k$.

Finally, we note that, as a consequence of the commutation relations (35) and (39), one can show that the mode functions $\{\varphi(k,\tau), \varphi^*(k,\tau)\}$, that represent the two independent solutions of (40), must satisfy the following Wronskian condition [10]

$$W[\varphi(k,\tau), \varphi^*(k,\tau)] \equiv \varphi(k,\tau)\frac{d}{d\tau}\varphi^*(k,\tau) - \Big(\frac{d}{d\tau}\varphi(k,\tau)\Big)\varphi^*(k,\tau) = \frac{i}{a^2}\,. \tag{41}$$

### 1. The spectrum

In cosmology the primary object of interest is the spectrum $\mathcal{P}$, which can be easily extracted from the equal time two-point function (correlator). The spectrum is a measure of the size of field fluctuations. Since each momentum mode evolves independently, and its amplitude is centered at zero, the spectum can be viewed as the covariance of all the modes with momenta in a thin shell, whose thickness is equal when viewed in logarithmic intervals of the momentum.

For example, for the scalar field $\hat{\varphi}$ the equal time correlator is related to the spectrum as follows,

$$\begin{aligned}\langle\Omega|\hat{\varphi}(\vec{x},\tau)\hat{\varphi}(\vec{x}\,',\tau)|\Omega\rangle &= \int\frac{d^3k}{(2\pi)^3}|\varphi(k,\tau)|^2 e^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\,, \qquad (r=\|\vec{x}-\vec{x}\,'\|)\\ &\equiv \int\frac{dk}{k}\mathcal{P}_\varphi(k,\tau)\frac{\sin(kr)}{kr}\,,\end{aligned} \tag{42}$$

where the scalar field spectrum is,

$$\mathcal{P}_\varphi(k,\tau)=\frac{k^3}{2\pi^2}|\varphi(k,\tau)|^2\,. \tag{43}$$

Now from Eqs. (23), (28) and (25) is follows that, in the zero curvature gauge, the curvature perturbation spectrum,

$$\langle\Omega|\hat{w}_\psi(\vec{x},\tau)\hat{w}_\psi(\vec{x}\,',\tau)|\Omega\rangle=\int\frac{dk}{k}\mathcal{P}_{w_\psi}(k,\tau)\frac{\sin(kr)}{kr}\,, \tag{44}$$

can be related to the spectrum of scalar field fluctuations as follows,

$$\mathcal{P}_{w_\psi}(k,\tau)=\frac{H^2}{\dot{\phi}^2}\mathcal{P}_\varphi(k,\tau)\,. \tag{45}$$

This means that, in order to get the spectrum of curvature perturbation $w_\psi$, whose effects can be measured today, one has to determine the spectrum of scalar field fluctuations during inflation, which is what we do next.

### 2. Scalar field spectrum in de Sitter inflation

But rather than determining the scalar field spectrum in general inflation (which is hard), we shall use a trick and first calculate the spectrum in de Sitter inflation (which is easy), promote the relevant parameters on which the spectrum depends to slowly varying functions of time (adiabatic or slow roll approximation), based in which we shall finally determine the spectrum of the scalar field and untimately of the curvature perturbation (again in slow roll approximation).

To begin, recall that in de Sitter inflation, $a=-1/(H_0\tau)$, such that $a''/a=2/\tau^2$, where $H_0$ is the (constant) Hubble parameter of de Sitter space, and $\tau<0$ is conformal time. With this we can write

Eq. (40) in de Sitter space as,

$$\Big(\frac{d^2}{d\tau^2}+k^2-\frac{2}{\tau^2}\Big)(a\varphi(k,\tau))=0\,. \tag{46}$$

Notice that we have dropped the terms in (40) containing $h_{00}$ and $h_{0i}$ and $\phi'$ terms. This can be justified in de Sitter space as follows. Firstly, the term containing $V''$ is proportional to the second slow roll paramter, $V''\simeq\eta_V V/M_P^2\simeq 3\eta_V H^2$, and since in de Sitter space $\eta_V=0$, we are justifed to neglect it. Similarly, $V'=\sqrt{2\epsilon_V}V/M_P=\sqrt{2\epsilon_V}3H^2M_P$, where $\epsilon_V=(1/2)M_P^2(V'/V)^2$ is a slow roll parameter. Again, in de Sitter space, $\epsilon_V$ (which is, in slow roll approximation, equal to $\epsilon=-\dot{H}/H^2$) must be equal to zero in de Sitter space. Indeed, if $V'$ were not equal to zero, the field $\phi$ would roll down the potential, which would break the de Sitter symmetry). And finally, from Eq. (4) we infer, $\phi'=a\dot{\phi}=a\sqrt{2\epsilon}M_PH$, which is also zero in de Sitter space.

The two linearly independent and properly normalised solutions of (46) are,

$$\varphi(k,\tau)=\frac{1}{a\sqrt{2k}}\Big(1-\frac{i}{k\tau}\Big)\mathrm{e}^{-ik\tau}\,;\qquad \varphi^*(k,\tau)\,. \tag{47}$$

As usual, normalisation of (47) is determined by the Wronskian (41). The Wronskian (41) does not determine the solutions uniquely however. Indeed, one can easily show that if (47) are properly normalised solutions, so are,

$$\varphi_{\rm gen}(k,\tau)=\alpha(k)\varphi(k,\tau)+\beta(k)\varphi^*(k,\tau)\,,\quad \varphi^*_{\rm gen}(k,\tau)\,, \tag{48}$$

where – in order not to change the Wronskian (41) – the complex constants $\{\alpha(k),\beta(k)\}$ must satisfy

$$|\alpha(k)|^2-|\beta(k)|^2=1\,. \tag{49}$$

This condition fixes one of the four real numbers, and thus does not uniquely specify the vacuum $|\Omega\rangle$ of the theory. For each $k$ there remain two arbitrary real numbers that are unspecified (the third number is an overall phase, that can be absorbed into the definition of the vacuum $|\Omega\rangle$ and thus has no physical relevance). There is one special choice out of all of these vacua, known as the Bunch-Davies vacuum, for which

$$\alpha(k)=1\,;\qquad \beta(k)=0\,;\qquad (\forall\vec{k}\,)\,, \tag{50}$$

which in fact corresponds to (47). An argument in favour of this choice is that, when one considers asymptotic past $\tau\to-\infty$, then the physical momentum $k/a\gg H$ decouples from the Universe's expansion, and thus it would cost a lot of energy to put any quantun into the state [11]. Moreover, for these states one can show that the energy in the field excitations, $\mathcal{E}_\varphi(k,\tau)\propto[|\alpha(k)|^2+|\beta(k)|^2]$, which is clearly minimised in the BD vacuum (50). This expression for the energy per mode is correct only in the adiabatic regime, in which $k/a\gg H$, and fails in the infrared ($k/a\leq H$), where the field couples

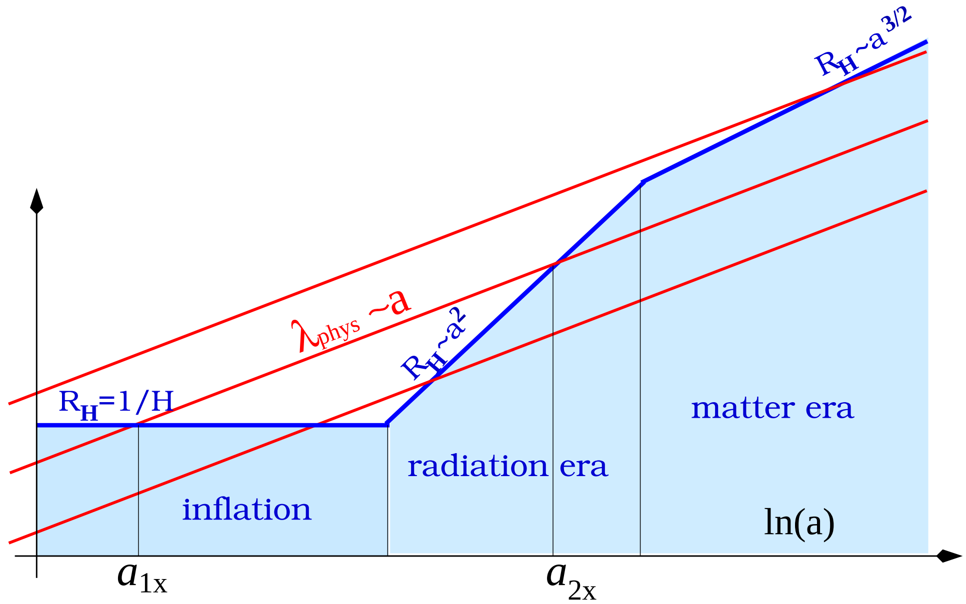


FIG. 2: The evolution of physical scales during inflation, radiation and matter era. Quantum fluctuations are generated at a scale $M$, probably smaller than the Planck scale $M_P$, by an unknown mechanism. During inflation the physical scale of the fluctuations grows with the Universe expansion, $\lambda_{\rm phys} \propto a$, while the Hubble scale remains approximately constant, $R_H = 1/H \simeq$ const., such that the scale of quantum fluctuations becomes larger than the Hubble radius after the first Hubble crossing, $a = a_{1x}$. While the amplitude of fluctuations on sub-Hubble scales, $R < 1/H$, scales as, $\varphi|_R \simeq 1/R$, it freezes out at super-Hubble scales, $\varphi|_R \simeq H/2\pi$ ($R \gg 1/H$), and remains approximately constant until the second horizon crossing, $a = a_{2x}$, in radiation or matter era, $\lambda_{\rm phys} \simeq 1/H$, at which the modes enter the Hubble radius and begin again oscillating.

strongly to gravity and where there is no reason to expect that (50) should be the correct choice. Worse still, as we will see below, the BD choice (50) results in a logarithmic infrared divergence for the equal time correlator (42) which renders the BD vacuum (50) unphysical in the deep infrared. We shall not bother to try to resolve this difficulty here, since it is not important for these lecture notes.

Instead we adopt the mode functions (47) as the solutions of (46), from which we can easily calculate the spectrum (43),

$$\mathcal{P}_\varphi(k,\tau) = \frac{k^2}{4\pi^2 a^2}\Big(1 + \frac{1}{(k\tau)^2}\Big) = \frac{H_0^2}{4\pi^2}\Big(1 + \frac{k^2}{(H_0 a)^2}\Big)\,, \tag{51}$$

where $H_0 = \dot{a}/a$ denotes the (constant) de Sitter space expansion rate. We have thus derived an important result: the spectrum for scalar field fluctuations in de Sitter space for a massless scalar field. But before we proceed, we pause and comment on the meaning of the result (51). Notice that, if one follows one comoving momentum, then it will in general quickly evolve from sub-Hubble scales $k/a \gg H$, on which the spectrum is that of a conformal vacuum, $\mathcal{P}_\varphi \simeq [\mathcal{P}_\varphi]_{\rm conf} = (k/[2\pi a])^2$, to super-Hubble scales, on which the spectrum is scale invariant and constant, $\mathcal{P}_\varphi \simeq H_0^2/(4\pi^2)$. This scale invariance of the spectrum is generic for massless scalars in de Sitter space, but it results in infrared problems for the field correlators, which can be seen from Eq. (42): the correlator is logarithmically divergent in the infrared. The solution to this malady of the state is to abandon (48), and instead choose the coefficients $\alpha(k)$ and $\beta(k)$ in (48) such that the vacuum creates infrared finite correlations. As long as one starts changing the vacuum on scales that correspond to today's super-Hubble scales, this will have little effects on the spectrum we observe today, and for that reason we shall not bother to discuss it further here. In passing we note that the deep infrared modes might have an effect on other observables, which include the three point correlator, also known as the bispectrum, and on which the Planck satellite has placed the strictest constraints up to now. Finally, the fact that the spectrum (51) 'freezes' very quickly (exponentially fast in comoving time) after the Hubble crossing,

$$\mathcal{P}_\varphi(k, \tau \to 0) \to \frac{H_0^2}{4\pi^2} \qquad (k/a \ll H_0)\,, \tag{52}$$

justifies neglecting the time dependent correction as long as we ask questions about deeply infrared modes, which is in fact the case for today's measurements. Having obtained the spectrum in de Sitter space, we can now promote it to the spectrum in quasi-de Sitter (*aka* slow roll) inflation as follows. When $H_0 \to H(t)$ is a slowly varying function of time, then

$$\mathcal{P}_{\varphi *} \simeq \frac{H_*^2(t)}{4\pi^2}\,, \qquad (k/a \simeq H) \tag{53}$$

represents the spectrum amplitude at the Hubble crossing (which we denote by $*$), *i.e.* for the modes for which $k/a \simeq H$. With this the following picture has emerged. Modes of Planckian or super-Planckian energy are generated by some unknown quantum process early in inflation. Their amplitude, which is dictated by the canonical commutation relation (35), then decays as $\varphi \propto k/a$ as the universe expands, until it reaches the Hubble length, when it freezes. As the modes expand further, their amplitude remains frozen on super-Hubble scales. This can be seen from the equation of motion for the modes with super-Hubble wavelengths already mentioned above,

$$(\partial_t + 3H)\partial_t \varphi(k, t) \simeq 0\,, \qquad (k/a) \ll H\,, \tag{54}$$

which as a solution contains a constant mode and a decaying mode, the former being the frozen mode. The evolution of the comoving modes (*i.e.* the modes with a constant physical wavelength) and of the Hubble radius with time is illustrated in figure 2. In this picture, different modes today will have different amplitude only because the Hubble rate at which they crossed the Hubble scale ($k/a = H$) was different. A detailed analysis shows that this way of calculating the amplitude of scalar fluctuations gives the correct answer for the spectrum within the inflationary slow roll paradigm. The deep reason for this agreement is the adiabaticity of $H(t)$. Figure 2 also shows how the modes, which cross the Hubble radius during inflation, become again sub-Hubble during the subsequent radiation or matter era, when they begin oscillating again.

The spectral amplitude of the curvature perturbation $w_\psi$ at Hubble crossing can be now read off from (45) and (53),

$$\mathcal{P}_{w_\psi *} = \frac{H_*^4}{\dot{\phi}_*^2(4\pi^2)} \,. \tag{55}$$

We are still not quite yet done with the calculation of the spectrum. Since $w_\psi$ is conserved on super-Hubble scales, the momentum dependence (spectral slope) at any given time arises as a consequnce of the time dependence of $H$ and $\dot{\phi}$. In other words, we can assume,

$$\mathcal{P}_{w_\psi}(k) = \mathcal{P}_{w_\psi *}\Big(\frac{k}{k_*}\Big)^{n_s-1} \,, \qquad \mathcal{P}_{w_\psi *} = \frac{H_*^4}{\dot{\phi}_*^2(4\pi^2)} \,, \tag{56}$$

where $k_* = (aH)_*$ is the momentum that corresponds to the first Hubble crosssing. Obviously, the spectrum amplitude at the Hubble crossing is given by (56). Since in adiabatic (slow roll) picture, the spectral slope is induced by the time dependence of the amplitude at the Hubble crossing (recall that once the Hubble radius is crossed, the amplitude of perturbations is frozen) the spectral slope $n_s - 1$ can be calculated from

$$n_s - 1 = \frac{d\ln[\mathcal{P}_{w_\psi}]}{d\ln(k)}\Big|_{k=Ha} = \frac{dt}{d\ln(Ha)}\frac{d\ln[H^4/\dot{\phi}^2]}{dt} = \frac{1}{(1-\epsilon)H}\Big[4\frac{\dot{H}}{H} - 2\frac{\ddot{\phi}}{\dot{\phi}}\Big] = -6\epsilon + 2\eta \,, \tag{57}$$

where we made use of $\epsilon = -\dot{H}/H^2$ and $\eta = -\ddot{\phi}/(H\dot{\phi}) + \epsilon$ and we dropped the terms that are higher order in slow roll parameters. With this we have now fully determined the spectrum of the curvature perturbation to leading order in slow roll parameters,

$$\mathcal{P}_{w_\psi}(k) = \mathcal{P}_{w_\psi *}\Big(\frac{k}{k_*}\Big)^{n_s-1} \,, \qquad \mathcal{P}_{w_\psi *} = \frac{H_*^2(t)}{8\pi^2\epsilon_* M_P^2} \,, \qquad n_s = 1 - 6\epsilon + 2\eta \,, \tag{58}$$

where, in order to get $\mathcal{P}_{w_\psi *}$ in that form, we made use of (4), according to which $\dot{\phi}^2 = 2\epsilon M_P^2 H^2$.

The derivation leading to the result (58) was based on several approximations, which seem reasonable. But in order to really be sure that the expression for the spectrum (58) is correct in slow roll approximation, one has to check it by performing a more rigorous analysis, and this what we do next. In what follows we perform a rigorous analysis of the inflationary spectrum of both scalar and tensor cosmological perturbations.

### C. Cosmological perturbations in inflation: a rigorous treatment*

This section is denoted by a star (∗). This means that it contains a supplemental material, which is meant for those who want to deepen their understanding of cosmological perturbation theory.

We shall begin by quoting the action for the tensor and scalar cosmological perturbations,

$$\mathcal{S}_{\rm graviton}[h_{ij}^{TT}] = \frac{M_P^2}{8}\int d^3x d\tau a^2\Big([(h_{ij}^{TT})']^2 - (\nabla h_{ij}^{TT})^2\Big) \tag{59}$$

$$\mathcal{S}_{\rm scalar}[w_\psi] = \int d^3x d\tau \frac{(az)^2}{2}\Big((w_\psi')^2 - (\nabla w_\psi)^2\Big)\,, \qquad z = \frac{\dot\phi}{H} = \frac{\phi'}{\mathcal{H}} = \sqrt{2\epsilon}M_P \times {\rm sign}[\dot\phi]\,, \tag{60}$$

where $h_{ij}^{TT}$ is the traceless transverse graviton defined in (14) and $w_\psi$ is the (Sasaki's) spatial curvature perturbation (18); equivalently one can define the Mukhanov field $v = azw_\psi$, whose canonical momentum is simple, $\pi_v = v'$. Since $z$ appears quadratically in (60), the sign$[\dot\phi]$ is unimportant and, from now on, we shall drop it from the analysis. Because of the two time derivatives, the fields $w_psi$ and $h_{ij}^{TT}$ constitute the (three) dynamical degrees of freedom of the theory. The remaining four (gauge invariant) fields (out of eight four can be removed by gauge fixing) are constraints that can be associated to the freedom of choosing the slicing $\Sigma \times \mathbf{R}$ of the space-time $\mathcal{M}$ (their gauge dependent cousins are $h_{00}$ and $h_{0i}$ that appear in the scalar action (29). One scalar is the gauge invariant lapse function, and the gauge invariant shift vector can be split into a longitudinal scalar and a transverse vector. The derivation of the actions (59–60) can be found, for example, in Refs. [3, 4]. The action for the graviton can be derived by first deriving the graviton action in flat space (which is easy), and then using the well known conformal transformation for the Ricci scalar to obtain the action in a conformally related space-time [5]. Cosmology can be considered as a special conformal transformation, since cosmological space-times are conformally related to Minkowski space, $g^b_{\mu\nu} = a^2(\tau)\eta_{\mu\nu}$.

### *1. Gravitons*

Let us now analyse the gravitons. From Eq. (59) it follows that gravitons in cosmology obey,

$$(\partial^2 + 2\mathcal{H}\partial_0)h_{ij}^{TT} = 0 \tag{61}$$

Notice that, when this equation is written in de Sitter space, it becomes identical to the equation of motion of a massless scalar in de Sitter (46). This fact has motivated many studies of quantum effects of massless scalars on de Sitter space, with the hope to learn something on the quantum effects of gravitons. In more general cosmological spaces the two equations differ however (*cf.* Eq. (31)). Of course, the action for the graviton also differs from that of a massless scalar, in that it is normalised differently, and in the tensor structure. This can be seen from the canonical momentum for the graviton,

$$\pi^{ij} = \frac{\delta\mathcal{S}_{\rm graviton}}{\delta(h_{ij}^{TT})'} = \frac{M_P^2}{4}a^2(h_{ij}^{TT})'\,, \tag{62}$$

where $\mathcal{S}_{\rm graviton}$ is given in (59). The proper canonical quantisation follows from the Dirac theory of constrained systems [6] (see also Ref. [3]),

$$[\hat{h}^{TT}_{ij}(\vec{x},\tau),\hat{\pi}^{kl}(\vec{x}\,',\tau)] = \frac{i}{2}\Big[P_{ik}P_{jl} + P_{il}P_{jk} - P_{ij}P_{kl}\Big]\delta^3(\vec{x}-\vec{x}\,') \tag{63}$$

where $P_{ij} = \delta_{ij} - \partial_i\partial_j/\nabla^2$ is the transverse projector. The complicated structure on the right hand side of (63) is necessary to assure the traceless and transverse conditions (14) of both $\hat{h}^{TT}_{ij}$ and $\hat{\pi}^{kl}$.

It is now convenient to decompose the graviton into Fourier modes as,

$$\hat{h}^{TT}_{ij}(x) = \frac{2}{M_P}\sum_{\alpha=+,\times}\int\frac{d^3k}{(2\pi)^3}\Big[\epsilon^{\alpha}_{ij}(k)h(k,\tau)\hat{a}_{\vec{k}\alpha}{\rm e}^{i\vec{k}\cdot\vec{x}} + \epsilon^{\alpha}_{ij}(k)^* h^*(k,\tau)\hat{a}^{+}_{\vec{k}\alpha}{\rm e}^{-i\vec{k}\cdot\vec{x}}\Big] \tag{64}$$

where $\hat{a}_{\vec{k}\alpha}$ and $\hat{a}^{+}_{\vec{k}\alpha}$ are the annihilation and creation operators ($\hat{a}_{\vec{k}\alpha}|\Omega\rangle = 0$), and

$$\Big[\hat{a}_{\vec{k}\alpha},\hat{a}^{+}_{\vec{k}'\alpha'}\Big] = (2\pi)^3\delta_{\alpha,\alpha'}\delta^3(\vec{k}-\vec{k}\,')\,, \qquad \Big[\hat{a}_{\vec{k}\alpha},\hat{a}_{\vec{k}'\alpha'}\Big] = 0\,, \qquad \Big[\hat{a}^{+}_{\vec{k}\alpha},\hat{a}^{+}_{\vec{k}'\alpha'}\Big] = 0\,, \tag{65}$$

and $\epsilon^{\alpha}_{ij}(\vec{k})$ ($\alpha = +,\times$) are the two graviton polarisation tensors, which characterise a massless spin two particle, and which obey,

$$\sum_{ij}\epsilon^{\alpha}_{ij}(\vec{k})\epsilon^{\alpha'}_{ij}(\vec{k})^* = \delta^{\alpha,\alpha'}\,, \qquad \sum_{\alpha}\epsilon^{\alpha}_{ij}(\vec{k})\epsilon^{\alpha}_{kl}(\vec{k})^* = \frac{1}{2}\Big[\bar{P}_{ik}\bar{P}_{jl} + \bar{P}_{il}\bar{P}_{jk} - \bar{P}_{ij}\bar{P}_{kl}\Big]\,, \tag{66}$$

where $\bar{P}_{ik} = \delta_{ik} - k^i k^j/k^2$ ($k = \|\vec{k}\,\|$) is the momentum space transverse projector.

The mode functions $h(k,\tau)$ are homogeneous, independent on polarisation $\alpha$, and obey (*cf.* Eq. (40))

$$\Big(\partial_0^2 + \vec{k}^2 - \frac{a''}{a}\Big)(ah(k,\tau)) = 0\,. \tag{67}$$

Unlike in the case of scalar perturbations discussed in subsection IV- B 2, we shall solve these equations for power law inflation, in which the scale factor,

$$a(\tau) = \Big((\epsilon-1)H_0\tau\Big)^{\frac{1}{\epsilon-1}}\,; \qquad H = H_0 a^{-\epsilon}\,, \tag{68}$$

where $\epsilon \ll 1$, and $\epsilon(\tau)$ is an adiabatic function of time, *i.e.* $\dot{\epsilon} \ll H\epsilon$, such that $\dot{\epsilon}$ is of higher order in slow roll parameters and can be neglected. With this in mind, we can write,

$$\frac{a''}{a} = \frac{2-\epsilon}{(1-\epsilon)^2}\frac{1}{\tau^2} + \mathcal{O}(\dot{\epsilon})\,. \tag{69}$$

such that Eq. (67) can be written as,

$$\Big(\frac{d^2}{d\tilde{\tau}^2} + 1 - \frac{2-\epsilon}{(1-\epsilon)^2}\frac{1}{\tilde{\tau}^2}\Big)(ah(k,\tau)) \simeq 0 \qquad (\tilde{\tau} = -k\tau)\,. \tag{70}$$

This is the familiar Bessel's differential equation, and the two linearly independent and properly normalised solutions can be written in terms of the Hankel functions of the first and second kind as,

$$h(k,\tau) = \frac{1}{a}\sqrt{\frac{-\pi\tau}{4}}H_\nu^{(1)}(-k\tau)\,, \qquad h^*(k,\tau) = \frac{1}{a}\sqrt{\frac{-\pi\tau}{4}}H_\nu^{(2)}(-k\tau)\,, \qquad \nu = \frac{3-\epsilon}{2(1-\epsilon)}\,. \tag{71}$$

Indeed, based on the Wronskian of the Hankel functions, $W[H_\nu^{(1)}(z), H_\nu^{(2)}(z)] = -4i/(\pi z)$, we see that the above solutions satisfy,

$$W[h(k,\tau), h^*(k,\tau)] = \frac{i}{a^2}\,, \tag{72}$$

which, in the light of Eqs. (63), (66) and (65), is the right Wronskian. Notice that, for $\epsilon > 0$, which is the usual condition in slow roll inflation, the graviton Bunch-Davies vacuum solutions (71) suffer from an analogous infrared malady as the de Sitter vacuum of a massless scalar field. The malady is cured by the analogous means: one has to choose the vacuum in the deep infrared such to deviate in an appropriate manner from the Bunch-Davies vacuum, and pick the graviton's mode coefficients $\alpha(k)$ and $\beta(k)$ such that $\int d^3k|h(k,\tau)|^2$ is rendered infrared finite. Even though these types of infrared problems are potentially serious, we shall not bother with discussing details of any such infrared regularisation procedure, and we shall continue with the analysis of the graviton correlator for the Bunch-Davies vacuum (71).

Analogous to Eqs. (42–43), we define the graviton spectrum $\mathcal{P}_{\rm graviton}$ as,

$$\begin{aligned}\langle\Omega|h_{ij}^{TT}(\vec{x},\tau)h_{kl}^{TT}(\vec{x}\,',\tau)|\Omega\rangle &= \frac{4}{M_P^2}\int\frac{d^3k}{(2\pi)^3}|h(k,\tau)|^2\sum_\alpha \epsilon_{ij}^\alpha(\epsilon_{kl}^\alpha)^* {\rm e}^{i\vec{k}\cdot(\vec{x}-\vec{x}\,')}\frac{\sin(kr)}{kr}\,, \qquad (r = \|\vec{x}-\vec{x}\,'\|) \\ &\equiv \int\frac{dk}{k}\mathcal{P}_{\rm graviton}(k,\tau)\frac{\sin(kr)}{kr}\frac{1}{4}\Big[\bar{P}_{ik}\bar{P}_{jl} + \bar{P}_{il}\bar{P}_{jk} - \bar{P}_{ij}\bar{P}_{kl}\Big]\end{aligned} \tag{73}$$

where

$$\mathcal{P}_{\rm graviton}(k,\tau) = \frac{4k^3}{\pi^2}\frac{|h(k,\tau)|^2}{M_P^2}\,. \tag{74}$$

We are primarily interested on super-Hubble scales, where the Hankel functions (of the first kind)

$$H_\nu^{(1)}(z) = \frac{1}{\sin(\pi\nu)}\Big({\rm e}^{i\pi\nu}J_\nu(z) - iJ_{-\nu}(z)\Big)\,, \qquad (|{\rm arg}[z]| < \pi) \tag{75}$$

(and similarly for $H_\nu^{(2)}(z)$) can be expanded as,

$$\begin{aligned}H_\nu^{(1)}(-k\tau) &= \frac{1}{\pi}\Big(-{\rm e}^{i\pi\nu}\Gamma(-\nu)\Big(\frac{-k\tau}{2}\Big)^\nu - i\Gamma(\nu)\Big(\frac{2}{-k\tau}\Big)^\nu\Big) + \mathcal{O}((-k\tau)^{\pm\nu+2}) \\ \nu &= \frac{3-\epsilon}{2(1-\epsilon)} \simeq \frac{3}{2} + \epsilon + \mathcal{O}(\epsilon^2)\,,\end{aligned} \tag{76}$$

where we made use of

$$J_\nu(z) = \Big(\frac{z}{2}\Big)^\nu\Big(\frac{1}{\Gamma(\nu+1)} - \frac{(z/2)^2}{\Gamma(\nu+2)} + \mathcal{O}(z^4)\Big)\,, \qquad (|{\rm arg}[z]| < \pi) \tag{77}$$

and

$$\frac{1}{\sin(\pi\nu)} = \frac{\Gamma(\nu)\Gamma(1-\nu)}{\pi} \, . \tag{78}$$

Since $\nu > 0$, the second term in (76) dominates for super-Hubble modes, for which $-k\tau \ll 1$. Upon inserting (76) into (74) gives for the graviton spectrum,

$$\mathcal{P}_{\rm graviton}(k,\tau) = \frac{H_0^2}{\pi^3 M_P^2} 2^{\frac{3-\epsilon}{1-\epsilon}} \Gamma^2\Big(\frac{3-\epsilon}{2(1-\epsilon)}\Big)(1-\epsilon)^{\frac{2}{1-\epsilon}}\Big(\frac{k}{H_0}\Big)^{-\frac{2\epsilon}{1-\epsilon}} \, . \tag{79}$$

To leading order in slow this expression becomes,

$$\mathcal{P}_{\rm graviton}(k,\tau) = \mathcal{P}_{\rm gr*}\Big(\frac{k}{k_*}\Big)^{n_{\rm gr}} \, , \qquad \mathcal{P}_{\rm gr*} = \frac{2H_0^2}{\pi^2 M_P^2}\Big[1 + 2\epsilon(1-\gamma_E - \ln(2))\Big] \, , \qquad n_{\rm gr} = -2\epsilon \, , \tag{80}$$

where $\gamma_E = -\psi(1) \simeq 0.57$ is the Euler constant and $\psi(z) = d\ln[\Gamma(z)]/dz$ is the di-gamma function. This result agrees with what would have obtained had we calculated the graviton spectrum by using the de Sitter space mode functions (47) in an analogous way as was done in subsection IV- B 2 [12]. What cannot be obtained by that procedure is the $\mathcal{O}(\epsilon)$ correction to the amplitude $\mathcal{P}_{\rm gr*}$ shown in (80). That correction can be obtained only by the more rigorous (slow roll) analysis presented in this subsection.

### *2. Scalars*

We shall now show how to calculate the spectrum of scalar cosmological perturbations from the action (60). We shall work within the slow roll inflationary paradigm, but otherwise make no further approximation. By varying the action (60) one easily gets the equation of motion and the canonical momentum,

$$w_\psi'' + 2\frac{(az)'}{az}\partial_0 w_\psi - \nabla^2 w_\psi = 0 \, , \qquad \pi_{w_\psi} = (az)^2 w_\psi' \, , \qquad (z = \dot{\phi}/H) \, , \tag{81}$$

from which the canonical quantisation follows,

$$[\hat{w}_\psi(\vec{x},\tau), \hat{\pi}_{w_\psi}(\vec{x}\,',\tau] = i\delta^3(\vec{x}-\vec{x}\,') \tag{82}$$

Our experience suggests to rewrite Eq. (81) for the rescaled (Mukhanov) field $\hat{v} \equiv (az)\hat{w}_\psi$,

$$\Big(\partial_0^2 - \nabla^2 - \frac{(az)''}{az}\Big)(az\hat{w}_\psi) = 0 \, . \tag{83}$$

The next step is to expand the field in Fourier modes,

$$\hat{w}_\psi = \int \frac{d^3k}{(2\pi)^3}\Big[{\rm e}^{i\vec{k}\cdot\vec{x}} w_\psi(k,\tau)\hat{b}_{\vec{k}} + {\rm e}^{-i\vec{k}\cdot\vec{x}} w_\psi^*(k,\tau)\hat{b}_{\vec{k}}^+\Big] \tag{84}$$

where the annihilation and creation operators obey,

$$\hat{b}_{\vec{k}}|\Omega\rangle = 0 \, , \qquad [\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}^+] = (2\pi)^3\delta^3(\vec{k}-\vec{k}') \, , \qquad [\hat{b}_{\vec{k}}, \hat{b}_{\vec{k}'}] = 0 \, , \qquad [\hat{b}_{\vec{k}}^+, \hat{b}_{\vec{k}'}^+] = 0 \, , \tag{85}$$

and the mode functions satisfy,

$$\Big(\partial_0^2 + k^2 - \frac{(az)''}{az}\Big)(azw_\psi(k,\tau)) = 0\,. \tag{86}$$

In order to solve this equation, we need to evaluate $(az)''/(az)$. To do that, we shall make use of the background equations (4), to arrive at,

$$\begin{aligned}(az)' &= a\frac{d}{dt}\Big[\frac{a\dot\phi}{H}\Big] = a^2\Big[\dot\phi + \frac{\ddot\phi}{H} + \frac{\dot\phi^3}{2M_P^2H^2}\Big] \\ \frac{(az)''}{az} &= \frac{\frac{d}{dt}[(az)']}{z} = (aH)^2\Big[2 + \Big(\frac{\dot\phi^2}{M_P^2H^2} + \frac{3\ddot\phi^2}{\dot\phi H}\Big) + \Big(\frac{\dot\phi^4}{2M_P^4H^4} + \frac{2\dot\phi\ddot\phi}{M_P^2H^3} + \frac{\dddot\phi}{\dot\phi H^2}\Big)\Big]\,. \end{aligned} \tag{87}$$

This can be expressed in terms of the 'slow roll' parameters,

$$\epsilon = -\frac{\dot H}{H^2}\,, \qquad \eta = -\frac{\ddot\phi}{H\dot\phi} + \epsilon\,, \qquad \xi_{(2)} = -\frac{\dddot\phi}{H^2\dot\phi}\,, \tag{88}$$

where the index in $\xi_{(2)}$ signifies that $\xi_{(2)}$ is a second order slow roll parameter. Making use of (68) and (88) we can rewrite (87) as,

$$\frac{(az)''}{az} = \frac{2 + (5\epsilon - 3\eta) + (6\epsilon^2 - 4\epsilon\eta - \xi_{(2)})}{(1-\epsilon)^2\tau^2} \tag{89}$$

In order to proceed we have to make an approximation. A useful approximation is to assume that the slow roll parameters change adiabatically in time (slow roll approximation). In this approximation Eq. (86) becomes Bessel's differential equation, and the canonically normalised mode functions can be expressed in terms of the Hankel functions (*cf.* Eqs. (67–71)),

$$w_\psi(k,\tau) = \frac{1}{az}\sqrt{\frac{-\pi\tau}{4}}H_\nu^{(1)}(-k\tau)\,, \qquad w_\psi^*(k,\tau) = \frac{1}{az}\sqrt{\frac{-\pi\tau}{4}}H_\nu^{(2)}(-k\tau)\,. \tag{90}$$

where

$$\nu^2 = \frac{(az)''}{az} + \frac{1}{4} = \frac{9 + (18\epsilon - 12\eta) + (25\epsilon^2 - 16\epsilon\eta - 4\xi_{(2)})}{4(1-\epsilon)^2}\,. \tag{91}$$

The spectrum (44) of the curvature perturbation $w_\psi = \hat\psi$ is then

$$\mathcal{P}_{w_\psi}(k,\tau) = \frac{H^2}{a^2\dot\phi^2}\frac{k^3|\tau|}{8\pi}\Big|H_\nu^{(1)}(-k\tau)\Big|^2\,, \tag{92}$$

which on super-Hubble scales, $(1-\epsilon)k|\tau| \ll 1$, yields,

$$\mathcal{P}_{w_\psi}(k,\tau) = \frac{2^{2\nu-3}\Gamma^2(\nu)}{\pi^3}\frac{H^2}{8\pi^2\epsilon M_P^2}\Big(\frac{k}{(1-\epsilon)Ha}\Big)^{3-2\nu} \tag{93}$$

where we made use of Eqs. (75–77). Now, when $\nu$ in (91) is expanded in powers of slow roll parameters, Eq. (93) yields the following expression for the scalar power spectrum,

$$\begin{aligned}\mathcal{P}_{w_\psi}(k,\tau) &= \mathcal{P}_{w_\psi *}\Big(\frac{k}{(1-\epsilon)aH}\Big)^{n_s-1} \\ \mathcal{P}_{w_\psi *} &= \frac{[1 + 2\epsilon(5 - 3\ln(2) - 3\gamma_E) - 2\eta(2 - \ln(2) - \gamma_E)]H^2}{8\epsilon\pi^2M_P^2} \\ n_s - 1 &= (-6\epsilon + 2\eta) + \frac{2}{3}\Big(-13\epsilon^2 + 4\epsilon\eta + \eta^2 + \xi_{(2)}\Big)\,. \end{aligned} \tag{94}$$

When compared with Eq. (58), we see that the amplitudes agree at the leading order $\mathcal{O}(1/\epsilon)$ and that the spectral slope $n_s - 1$ also agrees at the leading order $\mathcal{O}(\epsilon, \eta)$ in slow roll parameters. The result (94) is more general than (58) in that it also contains the next to leading order corrections in slow roll paramters. Notice, in particular, the weak additional time dependence of the amplitude in (94), which enters through $(aH)^{1-n_s} \approx (aH)^{6\epsilon-2\eta}$, whose origin can be traced to the fact that the physical momentum at the Hubble crossing, $k/a \simeq H$, is time dependent. This weak time dependence of $\mathcal{P}_{w_\psi}$ is absent in the approximate solution (58).

A rather complex analysis of temperature fluctuations of the CMB establishes a connection between the primordial graviton and scalar potentials spectra and the observed temperature fluctuations. The most prominent (and simplest) effect is the Doppler effect (see Part I) exhibited by photons as they move towards us and climb out of potential wells generated by the curvature perturbation $w_\psi$ at the last scattering surface. This effect is also known as the Sachs-Wolfe effect, firstly described in 1968.

According to the recent Planck analysis, the Planck data in conjunction with some other large scale observations (see Ade et al [Planck collaboration] XXII. Constraints on inflation [arXiv:1303.5082]) yield:

$$\begin{aligned}
\mathcal{P}_{w_\psi *} &= \frac{H_*^2}{8\epsilon_* \pi^2 M_P^2} = 2.441 \pm 0.092 \times 10^{-9} \\
n_s &= 0.9603 \pm 0.0073 \quad (65\%\ \text{C.L.}) \\
r &\equiv \frac{\mathcal{P}_{\text{graviton}*}}{\mathcal{P}_{w_\psi *}} < 0.11\,,
\end{aligned} \tag{95}$$

with today's fiducial comoving momentum $k_* = 0.002\ \text{Mpc}^{-1}$. This represents a more than five standard deviation detection of deviation from scale invariance, representing a strong support to the inflationary origin of cosmological perturbations. Together with un upper bound on the tensor-to-scalar ratio, $r < 0.11$, the spectral index in (95), rules out or disfavours many single field inflationary models (including all chaotic models with a $\phi^n$ $(n > 1)$ potential). However, hybrid inflationary models and single scalar with a non-minimally coupled inflaton (such as Higgs inflation), as well as the original 1980 Starobinsky's model with an $R^2$ term, are still in perfect agreement with observations.

### D. From inflationary cosmological perturbations to temperature fluctuations

The main results of the previous analyses are the spectra of the comoving curvature perturbation (58), (94) and of the graviton (80) produced by slow roll inflation. In postinflationary epochs which can be characterised by a constant equation of state $\mathcal{P} = w\rho$, where $\mathcal{P}$ denotes the pressure of the cosmological fluid and $w =$ const. is the equation of state parameter ($w = 1/3$ in radiation era, $w \approx 0$ in matter era) both the graviton and the curvature perturbation obey the simple equation,

$$\Big(\partial_0^2 - \nabla^2 - \frac{a''}{a}\Big)(a w_\psi) = 0\,, \qquad \Big(\partial_0^2 - \nabla^2 - \frac{a''}{a}\Big)(a h_{ij}^{TT}) = 0\,. \tag{96}$$

This equation becomes especially simple in radiation era, where $a \propto \tau$, such that $a''/a = 0$, and the solutions are

$$w_\psi(k,\tau) = \frac{w_{\psi 0}(k)}{ak}\sin(k\tau) + \frac{\bar{w}_{\psi 0}(k)}{a}\cos(k\tau)\,, \quad h_{ij}^\alpha(k,\tau) = \epsilon_{ij}^\alpha(k)\Big[\frac{h_0(k)}{ak}\sin(k\tau) + \frac{\bar{h}_0(k)}{a}\cos(k\tau)\Big]\,, \tag{97}$$

where $\tau_0$ denotes conformal time at the end of inflation. The amplitudes $w_{\psi 0}(k)$, $\bar{w}_{\psi 0}(k)$, $h_0(k)$ and $\bar{h}_0(k)$ are fixed by continuously matching $w_\psi(k,\tau)$ and $h(k,\tau)$ and at the end of inflation. For obvious reasons, the first terms in Eq. (97) are known as the growing solutions, while the second terms are the decaying solutions. Notice that on super-Hubble scales ($k\tau < 1$) the 'growing' modes have approximately constant amplitude, while the decaying modes decay as $\propto 1/a$, and soon become oblivious. If one is to represent the spectra $\mathcal{P}_{w_\psi}$ and $\mathcal{P}_{\rm graviton}$ by the amplitudes in (97), a good way of thinking about it is in the spirit of (classical) statistical field theory [13]. This means that $h_0(k)$ and $w_{\psi 0}(k)$ (for each $\vec{k}$) should be drawn from a Gaussian probability distribution,

$$P_h(h_0(k)) \propto \exp\Big[-\frac{|h_0(k)|^2}{2\sigma_h(k)}\Big]\,, \qquad P_{w_\psi}(w_{\psi 0}(k)) \propto \exp\Big[-\frac{|w_{\psi 0}(k)|^2}{2\sigma_{w_\psi}(k)}\Big]\,, \tag{98}$$

with the variances given by $\sigma_h = \langle |h_0(k)|^2\rangle$ and $\sigma_{w_\psi} = \langle |w_{\psi 0}(k)|^2\rangle$, respectively. It is clear from these equations that the phases of $h_0(k)$ and $w_{\psi 0}(k)$ play no role in the definition of the spectrum, and hence they are assumed to be randomly distributed. These initial conditions are known as the adiabatic initial conditions for cosmological perturbations. The statistical nature of their amplitude can be traced back to the uncertainty in the amplitude of each of the quantum oscillators early in inflation.

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.......................TO BE CONTINUED...........................

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[2] E. Komatsu *et al.*, "Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Interpretation," arXiv:1001.4538 [astro-ph.CO].

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[7] The Sasaki-Mukhanov field can be found in literature in different disguises, as the Mukhanov variable $v_M = a\hat{\varphi}$, as the Sasaki variable, $\hat{\psi} = (H/\dot{\phi})\hat{\varphi}$, *etc.* All of these variables differ by a time dependent rescaling, and are therefore physically equivalent.

[8] In the Adler-Deser-Misner (ADM) formalism, the freedom of choosing time is represented in the freedom of choosing a lapse function $N(x)$. In addition, there is a freedom to choose a shift vector $N^i(x)$, which, at the linear level, corresponds to the freedom of freely choosing the spatial components of $\xi^i$.

[9] Before gauge fixing, at the quadratic level $\varphi$ couples to the three scalars: $h_{00}$, the longitudial part of $h_{0i}$ and the trace of $h_{ij}$. Since the coordinate shift $\xi^\mu$ contains only two scalars ($\xi^0$ and the longitudinal part of $\xi^i$), it is not possible to completely remove the terms that couple $\varphi$ to the gravitational scalars. As we will see in section C, this coupling changes the equation of motion for $\varphi$.

[10] The Wronskian can be viewed as the normalisation of the scalar product for the wave function, $(\varphi,\varphi) = 1$, see *e.g.* Birrell and Davies [1].

[11] A state with $\beta(k) \neq 0$ is a pure state that contain $n(k) = |\beta(k)|^2$ quanta of the field $\varphi$ in a coherent superposition (for a more detailed explanation see the book of Birrell and Davies).

[12] The reader can easily check that such a calculation would result in,

$$\mathcal{P}_{\rm gr*} = \frac{2H_*^2}{\pi^2 M_P^2}\,, \qquad n_{\rm gr} = \frac{d\ln(\mathcal{P}_{\rm gr*})}{d\ln(k)}\bigg|_{k=Ha} = -\frac{2\epsilon}{1-\epsilon}\,,$$

which agrees with (74–80).

[13] Strictly speaking the amplified vacuum correlators that we discussed in previous sections are quantum in the sense that each field mode satisfies the minimum uncertainty relation. Nevertheless, quantum properties of super-Hubble modes can be ignored for most purposes in cosmology. A simple (but not rigorous) argument supporting this statement goes as follows. Consider, for example, the quantum scalar perturbation, and decompose it into Fourier modes, $\hat{\psi}(\vec{x},\tau) = \int[d^3k/(2\pi)^3]\hat{\psi}(\vec{k},\tau)\exp(i\vec{k}\cdot\vec{x})$, such that $[\hat{\psi}(\vec{k},\tau),(a\dot{\phi}/H)\hat{\psi}'(\vec{k}\,',\tau)] = (2\pi)^3\delta^3(\vec{k}-\vec{k}\,')$. Then for $k \ll aH$, $\{\hat{\psi}(\vec{k},\tau),\hat{\psi}(\vec{k}\,',\tau)\} \gg [\hat{\psi}(\vec{k},\tau),\hat{\psi}(\vec{k}\,',\tau)]$. In other words, the corresponding mode functions satisfy, $(\dot{\phi}/H)^2|\hat{\psi}(k,\tau)\partial_\tau\hat{\psi}(k,\tau)| \gg 1$, which means that the quantum properties of the field are suppressed when compared with the (classical) statistical properties. This suggests that the error made by replacing the rigorous density matrix description of the

state by a probability distribution of the corresponding classical field amplitude should lead to reasonably accurate answers to any cosmological measurement. A more complete treatment of the question, why the cosmological perturbation appear classical, even though their origin is quantum, is beyond the scope of these notes. Such a treatment would have to establish the decoherent agent that decoheres quantum cosmological perturbations either during inflation or subsequent epochs, and thus makes them approximately classical.