Classical field theory 2013 (NS-364B) – Supplementary lecture notes, by Tomislav Prokopec

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1. Wave equation

The free Maxwell equations are,

$$-\frac{1}{c}\partial_t\vec{E}+\nabla\times\vec{B}=0\,,\qquad \frac{1}{c}\partial_t\vec{B}+\nabla\times\vec{E}=0\,. \tag{1}$$

Each of these fields can be decomposed into the transverse and longitudinal components as follows,

$$\vec{E}=\vec{E}^T+\vec{E}^L\,,\qquad \vec{B}=\vec{B}^T+\vec{B}^L\,, \tag{2}$$

where

$$\vec{E}^L=\nabla^{-2}[\nabla(\nabla\cdot\vec{E})]\,,\qquad \vec{B}^L=\nabla^{-2}[\nabla(\nabla\cdot\vec{B})]\,.$$

Here ∇^{-2} is the inverse of the Laplacian operator ∇^2, defined by the corresponding Green function,

$$\nabla^2_{\vec{x}}G(\vec{x};\vec{x}\,')=\delta^3(\vec{x}-\vec{x}\,')=\nabla^2_{\vec{x}\,'}G(\vec{x};\vec{x}\,')\,. \tag{3}$$

The (vacuum) translationally and rotationally invariant solution of this equation is

$$G(\vec{x};\vec{x}\,')=-\frac{1}{4\pi}\frac{1}{\|\vec{x}-\vec{x}\,'\|}\,, \tag{4}$$

where by translational invariance we mean the invariance under $\vec{x}\to\vec{x}+\vec{d}$, where $\vec{d}$ is an arbitrary shift vector, and by rotational invariance, we mean the invariance under $\vec{x}\to R\cdot\vec{x}$, where $R\in O(3)$ is an orthogonal matrix, such that $R\cdot R^T=I=R^T\cdot R$.
The transverse components of equations (1) can be combined to yield the wave equations for the (transverse components of the) electric and magnetic fields,

$$-\partial^2\vec{E}^T(x)=0\,,\qquad -\partial^2\vec{B}^T=0\,,\qquad \partial^2=\eta^{\mu\nu}\partial_\mu\partial_\nu=\frac{1}{c^2}\partial_t^2-\nabla^2\,,\qquad \nabla^2=\sum_{i=1}^{3}\partial_i^2\,, \tag{5}$$

where here $(x)=(x^\mu)=(ct,\vec{x})$. These equations are linear differential equations, and the general solution can be found by performing a Fourier transformation. Making the ansatz

$$\vec{E}(x)=\int\frac{d^4k}{(2\pi)^4}\mathrm{e}^{ik\cdot x}\tilde{\vec{E}}(k^\mu)\,, \tag{6}$$

Eq. (5) reduces to the algebraic equation,

$$(k_\mu k^\mu)\tilde{\vec{E}}(k^\mu)=0,\qquad k^2=\eta_{\mu\nu}k^\mu k^\nu=k_0^2-\|\vec{k}\|^2\,. \tag{7}$$

The general solution to this equation is proportional to $\delta(k_\mu k^\mu)$, and can be written as,

$$\tilde{\vec{E}}(k^\mu)=2\pi\vec{E}_+(\vec{k})\delta(k_0-\omega/c)+2\pi\vec{E}_-(\vec{k})\delta(k_0+\omega/c)\,, \tag{8}$$

where the delta functions determine the *dispersion relation* for photons. Indeed, $\delta(k_\mu k^\mu) = (1/2\|\vec{k}\|)[\delta(k_0 - \|\vec{k}\|) + \delta(k_0 + \|\vec{k}\|)]$ tell us that the photon wave can have two frequencies,

$$k_0 = \pm\|\vec{k}\| = \pm\frac{\omega}{c}\,, \tag{9}$$

where the two signs refer to the positive and negative frequency poles. Thus, $\vec{E}_+(\vec{k})$ is the amplitude of the photon electric field projected onto the positive frequency pole $k_0 = \omega/c = \|\vec{k}\| \equiv k$, while $\vec{E}_-(\vec{k})$ is the amplitude of the photon electric field at the negative frequency pole, $k_0 = -\omega/c = -k$. When (8) is inserted into (6) and upon integration over k_0, one gets,

$$\vec{E}(x) = \int \frac{d^3k}{(2\pi)^3}\left[\mathrm{e}^{i(\omega t - \vec{k}\cdot\vec{x})}\vec{E}_+(\vec{k}) + \mathrm{e}^{-i(\omega t + \vec{k}\cdot\vec{x})}\vec{E}_-(\vec{k})\right], \qquad \nabla\cdot\vec{E} = 0\,. \tag{10}$$

The last condition imposes transversality. By the analogous procedure, one can construct the general wave solution for the magnetic field,

$$\vec{B}(x) = \int \frac{d^3k}{(2\pi)^3}\left[\mathrm{e}^{i(\omega t - \vec{k}\cdot\vec{x})}\vec{B}_+(\vec{k}) + \mathrm{e}^{-i(\omega t + \vec{k}\cdot\vec{x})}\vec{B}_-(\vec{k})\right], \qquad \nabla\cdot\vec{B} = 0\,. \tag{11}$$

The exponential factors in (10–11), $\Phi_\pm(\vec{x},t) = \omega t \mp \vec{x}\cdot\vec{k}$, are the phases of a wave with a wavevector $\vec{k}$. By demanding a stationary phase, $d\Phi_\pm(\vec{x},t) = \omega dt \mp (d\vec{x})\cdot\vec{k} = 0$, one gets the phase velocities,

$$v_\pm = \frac{d\vec{x}}{dt} = \pm\frac{\omega}{k}\hat{k} = \pm c\hat{k}\,, \qquad \hat{k} = \frac{\vec{k}}{k}\,. \tag{12}$$

This tells us that both phase speeds are equal to the speed of light, $\|\vec{v}_\pm\| = c$ and that wave crests of positive frequency waves move in the direction of $\vec{k}$, $\vec{v}_+ \propto +\vec{k}$, while wave crests of negative frequency waves move in the direction opposite to $\vec{k}$, $\vec{v}_- \propto -\vec{k}$.
Notice that the transversality conditions, $\nabla\cdot\vec{E} = \nabla\cdot\vec{B}$, become very simple in momentum space, $\vec{k}\cdot\vec{E}_\pm(\vec{k}) = 0 = \vec{k}\cdot\vec{B}_\pm(\vec{k})$. This means that electromagnetic waves are orthogonal on the direction of motion. Furthermore, when the free Maxwell's equations (1) are written in momentum space, one finds

$$\mp i\frac{\omega}{c}\vec{E}_\pm(\vec{k}) - i\vec{k}\times\vec{B}_\pm(\vec{k}) = 0\,, \qquad \pm i\frac{\omega}{c}\vec{B}_\pm(\vec{k}) - i\vec{k}\times\vec{E}_\pm(\vec{k}) = 0\,, \tag{13}$$

or

$$\vec{E}_\pm(\vec{k}) = \mp\hat{k}\times\vec{B}_\pm(\vec{k})\,, \qquad \vec{B}_\pm(\vec{k}) = \pm\hat{k}\times\vec{E}_\pm(\vec{k})\,. \tag{14}$$

From these relations we see that $\|\vec{E}_\pm\| = \|\vec{B}_\pm\|$ and that the three vectors $\{\vec{k},\vec{E}_+,\vec{B}_+\}$ ($\{\vec{k},\vec{E}_-,\vec{B}_-\}$) make up a positively (negatively) oriented orthogonal system. Recall that three unit vectors $\{\hat{e}_{(1)},\hat{e}_{(2)},\hat{e}_{(3)}\}$ of a postively oriented Cartesian orthogonal coordinate system satisfy, $\hat{e}_{(3)} = \hat{e}_{(1)}\times\hat{e}_{(2)}$.

2. Green functions

We shall show how to construct the retarded Green function (also known as the retarded propagator) for a real massless scalar field ϕ, whose action is given by

$$S_\phi[\phi] = \int d^4x\Big(\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu}\Big)\,. \tag{15}$$

The retarded Green function for this theory obeys the equation of motion,

$$-\partial_x^2 G_r(x;x') = \delta^4(x-x') = -\partial_{x'}^2 G_r(x;x')\,,\qquad \partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{16}$$

The retarded green function is useful since it allows us to solve a general equation of the form,

$$-\partial^2\phi(x) = j_\phi(x)\,. \tag{17}$$

Indeed, the solution is simply,

$$\phi(x) = \int d^4x' G_r(x;x')j_\phi(x')\,. \tag{18}$$

This solution is consistent with causality. Namely it has the property of retardation, which means that $\phi(x)$ can be influenced by events at x' which lie within (and on) the past light cone of x.
We seek the (vacuum) translationally and rotationally invariant solution of Eq. (16), which means that our solution will be of the form $G_r = G_r(x-x')$. With this simplification, it is advantageous to perform a 4-dimensional Wigner transform (defined as the Fourier transform with respect to the relative coordinate $x-x'$),

$$G_r(x-x') = \int \frac{d^4k}{(2\pi)^4} \mathrm{e}^{-\imath k\cdot(x-x')}\tilde{G}_r(k^\mu)\,. \tag{19}$$

Since $\delta^4(x-x') = \int [d^4k/(2\pi)^4]\mathrm{e}^{-\imath k\cdot(x-x')}1$, Eq. (16) simplifies to,

$$(k_\mu k^\mu)\tilde{G}_r(k^\mu) = 1\,, \tag{20}$$

whose general solution can be written as,

$$\tilde{G}_r(k^\mu) = \frac{1}{k_\mu k^\mu} + 2\pi F(k^\mu)\delta(k_\mu k^\mu)\,. \tag{21}$$

The first part in (21) is the particular solution, the second represents a homogeneous solution, in which $F(k^\mu)$ is an arbitrary function, except that when multiplied by $k_\mu k^\mu$ it must vanish 'on shell', *i.e.* $[F(k^\mu)(k_\mu k^\mu)]_{k_\mu k^\mu=0} = 0$ (otherwise (21) would not be a solution). We shall now see that different choices of the homogeneous part in (21) yield different Green functions. The retarded Green function corresponds to the choice of $F(k^\mu)$ such that $G_r = 0$ for $t < t'$, *i.e.* for t' in the future of t. This causal structure of the retarded Green function is not present in the original equation (21), but instead it is imposed as the physical requirement on the solution. For example, the homogeneous part for the advanced Green function G_a ought to be chosed such that G_a vanishes when t' is in the past of t ($t' < t$). Before we proceed to integrating over the momenta, note that Eq. (21) can be written as,

$$\tilde{G}_r(k^\mu) = \frac{1}{2k}\left(\frac{1}{k_0-k} - \frac{1}{k_0+k}\right) + \frac{\pi}{k}\Big(F_+(\vec{k})\delta(k_0-k) + F_-(\vec{k})\delta(k_0+k)\Big)\,,\qquad (k = \|\vec{k}\,\|)\,, \tag{22}$$

where $F_\pm(\vec{k}) = F(k_0 = \pm\|\vec{k}\|, \vec{k})$. When this is inserted into Eq. (19), and one integrates over k_0, the first part in (22) will generate the principal part (pp) of the integral over the poles of the Green function, while the second part the homogeneous part (hp). The k_0 integral over the principal part can be evaluated as follows,

$$\begin{aligned}\int\frac{dk_0}{2\pi}[\tilde{G}_r(k^\mu)]_{pp} &= \frac{\Theta(\Delta t)}{4\pi k}\left(\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})}-\mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})}\right)\int_{-\infty}^{+\infty}\frac{dy}{y}\mathrm{e}^{-\imath y}\\ &+ \frac{\Theta(-\Delta t)}{4\pi k}\left(\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})}-\mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})}\right)\int_{+\infty}^{-\infty}\frac{dy}{y}\mathrm{e}^{-\imath y}\,,\end{aligned} \tag{23}$$

where $y=(k_0\mp k)\Delta t$. The (principal value) integrals are easily evaluated,

$$\int_{-\infty}^{+\infty}\frac{dy}{y}\mathrm{e}^{-\imath y}=\int_{-\infty}^{+\infty}\frac{\cos(y)dy}{y}-\imath\int_{-\infty}^{+\infty}\frac{\sin(y)dy}{y}=0-\imath\pi\;;\qquad\int_{+\infty}^{-\infty}\frac{dy}{y}\mathrm{e}^{-\imath y}=0+\imath\pi\,, \tag{24}$$

such that Eq. (23) yields,

$$\int\frac{dk_0}{2\pi}[\tilde{G}_r(k^\mu)]_{pp} = -\frac{\imath}{4k}\left[\Theta(\Delta t)-\Theta(-\Delta t)\right]\left(\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})}-\mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})}\right)\,. \tag{25}$$

On the other hand, the homogeneous part of the Green function (22) yields

$$\int\frac{dk_0}{2\pi}[\tilde{G}_r(k^\mu)]_{hp}=\left(\frac{F_+(\vec{k})}{2k}\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})}+\frac{F_-(\vec{k})}{2k}\mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})}\right)\left[\Theta(\Delta t)+\Theta(-\Delta t)\right]\,, \tag{26}$$

where, for convenience, we added on the right a factor $1=[\Theta(\Delta t)+\Theta(-\Delta t)]$.
Now, by comparing Eq. (26) with (33) we see that

$$F_+(\vec{k})=-F_-(\vec{k})=-\frac{\imath}{2} \tag{27}$$

is the simplest choice for the homogeneous parts such that the resulting retarded Green function vanishes when $t'>t$, as it is required by causality. With this choice we get for the retarded homogeneous part

$$\int\frac{dk_0}{2\pi}[\tilde{G}_r(k^\mu)]_{hp,ret}=-\frac{\imath}{4k}\left(\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})}-\mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})}\right)\left[\Theta(\Delta t)+\Theta(-\Delta t)\right]\,. \tag{28}$$

Let us now compare this with the contribution that one obtains by integrating around the positive and negative frequency poles, $k_0=-\omega/c$ and $k_0=\omega/c$. According to the residue theorem, which states that an integral over a closed contour of a meromorphic function $f(z)$ (which is a function of a complex variable z that is analytic everywhere insider the contour $\mathcal{C}$, except perhaps at a finite number of points whether the function has finite poles) is given in terms of the sum over the residues contained inside the contour of integration $\mathcal{C}$ by,

$$\oint_{\mathcal{C}}f(z)dz=2\pi\imath\sum_i\mathrm{Res}[f,z_i]\,, \tag{29}$$

where z_i are the points where the function has poles, and the orientation of the contour $\mathcal{C}$ is counterclockwise. In this lecture notes we will deal only with simple poles, in which case the residues of f can be calculated as

$$\mathrm{Res}[f,z_i]=\lim_{z\to z_i}[(z-z_i)f(z)]\,,\qquad\text{(for } z_i \text{ a simple pole of } f(z)) \tag{30}$$

(for a more detailed account of complex analysis, see Arfken, Mathematical Methods for Physicists).

Let us now define the positive and negative frequency (vacuum) Wightman functions G^+ and G^-, respectively, as the homogeneous contributions to the Green function that arise from integrating around the positive and negative frequency poles. By convention G^+ is obtained by integrating $1/(k^\mu k_\mu)$ counterclockwise, while G^- by integrating clockwise around the corresponding pole. The result is

$$\begin{aligned}\tilde{G}^-(\vec{k},\Delta t) &= -2\pi\imath \mathrm{Res}\,[f, k_0 = -\omega/c] = \frac{\imath}{2k}\mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})} \\ \tilde{G}^+(\vec{k},\Delta t) &= 2\pi\imath \mathrm{Res}\,[f, k_0 = \omega/c] = \frac{\imath}{2k}\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})}\,,\end{aligned} \tag{31}$$

where

$$f = \frac{1}{2\pi(k_0-k)(k_0+k)}\mathrm{e}^{-\imath k\cdot r}\,.$$

Comparing (31) with (28) reveals that the homogeneous part of the retarded Green function must be chosen such to be equal to

$$\int\frac{dk_0}{2\pi}[\tilde{G}_r(k^\mu)]_{hp,ret} = \frac{1}{2}(\tilde{G}^- - \tilde{G}^+)\,. \tag{32}$$

Adding this to the particular part (33) yields the retarded Green function

$$\int\frac{dk_0}{2\pi}\tilde{G}_r(k^\mu) = -\frac{\imath\Theta(\Delta t)}{2k}\left[\mathrm{e}^{-\imath(\omega\Delta t-\vec{k}\cdot\vec{r})} - \mathrm{e}^{\imath(\omega\Delta t+\vec{k}\cdot\vec{r})}\right] = \Theta(\Delta t)[\tilde{G}^-(\vec{k},\Delta t) - \tilde{G}^+(\vec{k},\Delta t)]\,. \tag{33}$$

The terms multiplying the Heaviside theta function is also known as the Pauli-Jordan (or spectral) two point function,

$$G_{\mathrm{PJ}} = G^- - G^+\,. \tag{34}$$

Before we proceed further, let us now step back to understand better what we have so far learned. We have shown that the retarded Green function is obtained when the homogeneous part is chosen as follows (*see* Eqs. (22) and (27):

$$\tilde{G}_r(k^\mu) = \frac{1}{2k}\left(\frac{1}{k_0-k} - \frac{1}{k_0+k}\right) - \frac{\imath\pi}{2k}\left(\delta(k_0-k) - \delta(k_0+k)\right)\,. \tag{35}$$

Now, making use of the Plemelj-Sokhotski relation,

$$\frac{1}{x\mp\imath\epsilon} = \mathcal{P}\frac{1}{x} \pm \imath\pi\delta(x)\,, \tag{36}$$

where $\mathcal{P}$ denotes the principal part, Eq. (35) can be recast as,

$$\tilde{G}_r(k^\mu) = \frac{1}{2k}\left(\frac{1}{k_0-k+\imath\epsilon} - \frac{1}{k_0+k+\imath\epsilon}\right) = \frac{1}{(k_0+\imath\epsilon)^2-k^2}\,, \tag{37}$$

which tells us the contour integration prescription for the retarded Green function: both poles are shifted below the real axis:

$$(k_0)_{\mathrm{poles}} = \pm\frac{\omega}{c} - \imath\epsilon\,, \qquad \text{(retarded Green function)} \tag{38}$$

as it is claimed (without proof) in the lecture notes. This has now been proved. In physics (primarily in quantum field theory) different (Green) two-point functions turned out to be useful, which have different pole prescriptions, which is synonymous to different choices of the homogeneous parts of the solution. In particular, the Feynman (time ordered) and anti-Feynman (anti-time ordered) Green functions are used, whose pole prescription is $(k_0)_{\rm poles} = \pm[(\omega/c) - \imath\epsilon]$ ($(k_0)_{\rm poles} = \pm[(\omega/c) + \imath\epsilon]$). Another Green function that is often used is the advanced Green function, which solves the problem of evolution backwards in time. Not surprisingly, the advanced pole prescription is opposite to that for the retarded Green function in (38),

$$(k_0)_{\rm poles} = \pm\frac{\omega}{c} + \imath\epsilon\,, \qquad \text{(advanced Green function)}\,. \tag{39}$$

By making use of the Plemelj-Sokhotski relation, one can show that not all two point functions are independent and relations between different Green functions can be established.

In order to get the direct (coordinate) space expression for the retarded Green function, one still needs to perform integrations over the spatial momenta. In Gleb's lecture notes a procedure how to perform the integrals to obtain $G_r(x-x')$ is presented. Here we shall present a slightly different procedure. We shall firstly calculate the Wightman functions, and then, by making use of (32), we shall calculate the retarded Green function in direct space.

We shall perform the spatial momentum integrations in (19) in spherical coordinates for $\vec{k} : (k, \theta, \phi)$. Taking $\vec{r} = \vec{x} - \vec{x}'$ to point in the z-direction, one gets $\vec{k}\cdot\vec{r} = kr\cos(\theta)$, where $r = \|\vec{x} - \vec{x}'\|$ and $\int d^3k = 2\pi\int_0^\infty k^2 dk \int_{-1}^{+1} dx$, $x = \cos(\theta)$, such that the positive and negative frequency Wightman functions (31) become,

$$\begin{aligned}
G^+(r) &\stackrel{?}{=} \int_0^\infty \frac{k^2 dk}{4\pi^2}\int_{-1}^1 dx \mathrm{e}^{\imath krx}\frac{\imath}{2k}\mathrm{e}^{-\imath kc\Delta t} = \frac{\imath}{4\pi^2 r}\int_0^\infty dk\sin(kr)\mathrm{e}^{-\imath kc\Delta t} \\
G^-(r) &\stackrel{?}{=} \int_0^\infty \frac{k^2 dk}{4\pi^2}\int_{-1}^1 dx \mathrm{e}^{\imath krx}\frac{\imath}{2k}\mathrm{e}^{\imath kc\Delta t} = \frac{\imath}{4\pi^2 r}\int_0^\infty dk\sin(kr)\mathrm{e}^{\imath kc\Delta t}\,.
\end{aligned} \tag{40}$$

Both of these integrals contain integrands that are complex oscillatory functions with an amplitude that does not decrease as k increases, and hence they are not convergent. A way to evaluate the integrals is to promote $t - t'$ to a complex number (this procedure is called *analytic continuation* and it is of a tremendous importance in physics). It suffices to add a small imaginary part to $\Delta t = t - t'$ according to

$$\begin{aligned}
G^+(x-x') &= \frac{\imath}{4\pi^2 r}\int_0^\infty dk\frac{\mathrm{e}^{\imath kr} - \mathrm{e}^{-\imath kr}}{2\imath}\mathrm{e}^{-\imath k(\Delta t - \imath\epsilon)} \\
G^-(x-x') &= \frac{\imath}{4\pi^2 r}\int_0^\infty dk\frac{\mathrm{e}^{\imath kr} - \mathrm{e}^{-\imath kr}}{2\imath}\mathrm{e}^{\imath k(\Delta t + \imath\epsilon)}\,,
\end{aligned} \tag{41}$$

where $\epsilon > 0$ is an infinitesimal positive real number. The new integrals are convergent. We emphasise that it is of essential importance to keep these ϵ's until the end of the calculation, since they remind us in which part of the complex plane the two point functions were convergent. Furthermore, these direct space ϵ's should not be confused with the momentum space ϵ's: they have a different meaning and their origin is different. It is hence customary to keep the ϵ's in the expressions for the resulting Green functions, in order to serve as a reminder from what analytic continuation they originated.

The first integral in (41) yields the following positive frequency Wightman function,

$$
\begin{aligned}
G^{+}(x-x') &= \frac{1}{8\pi^2 r}\left(\frac{-1}{\imath[r-(\Delta t-\imath\epsilon)]}-\frac{-1}{-\imath[r+(\Delta t-\imath\epsilon)]}\right) \\
&= -\frac{\imath}{4\pi^2}\frac{1}{(\Delta t-\imath\epsilon)^2-r^2} \equiv -\frac{\imath}{4\pi^2}\frac{1}{\Delta x_+^2}\,,
\end{aligned} \tag{42}
$$

while the second integral yields the following negative frequency Wightman function,

$$
G^{-}(x-x') = -\frac{\imath}{4\pi^2}\frac{1}{(\Delta t+\imath\epsilon)^2-r^2} \equiv -\frac{\imath}{4\pi^2}\frac{1}{\Delta x_-^2}\,, \tag{43}
$$

where we defined the following complex 'distance' functions,

$$
\Delta x_-^2 = (t-t'+\imath\epsilon)^2 - \|\vec{x}-\vec{x}'\|^2\,, \qquad \Delta x_+^2 = (t-t'-\imath\epsilon)^2 - \|\vec{x}-\vec{x}'\|^2\,. \tag{44}
$$

What remains to be done is to make use of relations (33–34),

$$
G_r = \Theta(t-t')G_{\rm PJ}\,, \qquad G_{\rm PJ} = G^- - G^+ \tag{45}
$$

to construct the direct space retarded Green function. On order to do that, it is useful to rewrite the Wightman functions by making use of the Plemelj-Sokhotski relation (36)

$$
G^{+}(x-x') = -\frac{\imath}{4\pi^2}\left(\mathcal{P}\frac{1}{\Delta x_\epsilon^2} - \imath\pi{\rm sign}(\Delta t)\delta(\Delta x_\epsilon^2)\right)\,, \quad \Delta x_\epsilon^2 = (t-t')^2 - (\|\vec{x}-\vec{x}'\|^2+\epsilon^2)\,. \tag{46}
$$

Similarly, the negative frequency Wightman function can be recast as,

$$
G^{-}(x-x') = -\frac{\imath}{4\pi^2}\left(\mathcal{P}\frac{1}{\Delta x_\epsilon^2} - \imath\pi{\rm sign}(\Delta t)\delta(\Delta x_\epsilon^2)\right)\,. \tag{47}
$$

Now inserting these results into Eq. (45) we get,

$$
G_{\rm PJ}(x-x') = -\frac{{\rm sign}(\Delta t)}{2\pi}\delta(\Delta x_\epsilon^2)\,, \qquad G_r(x-x') = -\frac{\Theta(t-t')}{2\pi}\delta(\Delta x_\epsilon^2)\,. \tag{48}
$$

In these expressions ϵ appears quadratically and can be set to zero. This then results in the retarded Green function as derived in the lecture notes and lectures. In the lectures we have shown that the retarded Green function derived here for a scalar field theory can be also used in electromagnetism, where the equations to solve are the wave equations with sources,

$$
-\partial^2\vec{E}^T(x) = \frac{4\pi}{c^2}\partial_t\vec{j}^T(x)\,, \qquad -\partial^2\vec{B}^T(x) = -\frac{4\pi}{c}\nabla\times\vec{j}^T(x)\,. \tag{49}
$$

Equivalently, one can write the equation of motion for the transverse components of the vector potential,

$$
-\partial^2\vec{A}^T(x) = -\frac{4\pi}{c}\vec{j}^T(x)\,, \tag{50}
$$

with $\vec{E}^T = -\partial_t\vec{A}^T$ and $\vec{B}^T = \nabla\times\vec{A}^T$. The solutions are,

$$
\vec{A}^T(x) = \frac{1}{c}\int d^3x'\frac{\vec{j}^T(\vec{x}',t_r)}{\|\vec{x}-\vec{x}'\|}\,, \qquad t_r = t - \frac{\|\vec{x}-\vec{x}'\|}{c}\,, \tag{51}
$$

where, when integrating over time ($dx_0' = cdt'$), we used,

$$G_r(x-x') = -\frac{\Theta(t-t')}{4\pi r}\delta(c(t-t') - \|\vec{x}-\vec{x}\,'\|)\,. \tag{52}$$

We have thus shown that the source current $\vec{j}(\vec{x}\,', t')$ will contribute to building a potential at $(ct, \vec{x})$ if t' lies on the past light cone of $(ct, \vec{x})$, as it is consistent with causality. Since photons propagate with the speed of light, only the currents that lie on the past light cone contribute; because photons are massless, those that lie within the past light cone do not contribute in electromagnetism. In more general cases, which include scalar theories with a nonvanishing mass, also sources from within the past light cone generally contribute.

3. Dipole radiation

[under construction]

4. Solitons

A solitary wave is a phenomenon discovered in 1834 by John Scott Russell, when he saw a solitary wave being created by a boat that had rapidly stopped. Solitary waves persit and propagate with a constant speed for a long time without losing in strength. Russell himself called it 'the wave of translation', but the phenomenon came to be known as solitary waves. An important modern example are tsunamis, which can propagate across the whole Pacific ocean as high speed radial waves (tsunamies lose strength with the distance because they propagate radially, but gain in amplitude when they reach shallow sseas). Scott Russell observed solitary waves, but did not understand their physical origin. Today we know that a necessary condition for existence of solitary waves in a system are nonlinear (self-)interactions. *Solitons* are a special kind of solitary waves that have the property that they can collide against each other without getting perturbed. The deep reason for this is the mathematical property known as integrability, but we will not study it further here.
As an example, we shall construct a kink solution in a real scalar field theory in two space-time dimensions with a quartic selfinteraction. A configuration that resembles a solitary wave can be obtained by a superposition of two opposite kinks moving at the same speed in the same direction.
Let us begin our consideraction by writing down the action for a real scalar field ϕ. Its relativistically covariant form in D space-time dimensions is,

$$S_\phi = \int d^Dx\Big\{\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)\eta^{\mu\nu} - V(\phi)\Big\}\,, \tag{53}$$

where $\eta^{\mu\nu}$ is the (inverse) Minkowski metric, $\eta^{\mu\nu}\eta_{\nu\rho} = \delta^\mu_{\ \rho}$. The corresponding equation of motion results from the variation principle, $\delta S_\phi/\delta\phi(x^\mu) = 0$,

$$-\partial^2\phi(x^\mu) - \frac{dV}{d\phi} = 0\,, \qquad \partial^2 = \eta^{\mu\nu}\partial_\mu\partial_\nu\,. \tag{54}$$

The energy functional of a configuration $\phi = \phi(x^\mu)$ is then,

$$E[\phi] = \int d^{D-1}x\Big\{\frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\Big\}\,, \tag{55}$$

where $(\nabla\phi)^2 = \sum_{i=1}^{D-1}(\partial_i\phi)^2$.
Let us now for simplicity consider the case when $D = 2$ and take a static limit, in which $\phi = \phi(x)$ and $\partial^2 \to -(d/dx)^2$, such that (54) simplifies to

$$\frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}\,. \tag{56}$$

Multiplying this by $d\phi/dx$ yields,

$$\frac{d}{dx}\Big\{\frac{1}{2}\Big(\frac{d\phi}{dx}\Big)^2\Big\} = \frac{d}{dx}V(\phi)\,, \tag{57}$$

which, when integrated, gives

$$\Big(\frac{d\phi}{dx}\Big)^2 = 2V(\phi)\,, \tag{58}$$

where we ignored the constant of integration. This equation can be solved; the solution can be written implicitly as the integral,

$$\int_{\phi_0}^{\phi}\frac{d\phi'}{\sqrt{2V(\phi')}} = x - x_0\,, \qquad \phi_0 = \phi(x_0)\,. \tag{59}$$

Take as an example a quartic potential,

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - v_0^2)^2\,, \qquad v_0^2 = \frac{\mu^2}{\lambda}\,. \tag{60}$$

v_0 is a parameter that measures the strength of the field condensate (sometimes also refered to as the vacuum expectation value, or the *vev*), and has nothing to do with field velocity. Notice that this potential, and the corresponding action, is symmetric under $\phi \to -\phi$; this is known as the $\mathbb{Z}_2 = \{-1, +1\}$ discrete symmetry, and it has only two elements. In its lowest energy state (which is also known as the vacuum state), the scalar field acquires a condensate $\phi_{\rm vac,\pm} = \pm v_0$, which has a vanishing energy, $E[\phi_{\rm vac,\pm}] = 0$. These are the lowest energy states, or the vacuum states, of the thoery. This theory is said to exhibit the phenomenon known as *spontaneous symmetry breaking*, which states that the symmetry of the vacuum (in which $\phi = \pm v_0$) is lower than the symmetry of the underlying action. Namely, once a field condenses into say $\phi = \phi_- = -v_0$, the action will not be $\mathbb{Z}_2$-symmetric around that state. In this case the vacuum condensate breaks the symmetry of the action completely to I (identity). Notice that the set of all vacuum states, which is often denoted by $\mathcal{M}$, is equal to the the numer of states in the symmetry group of the original action, namely, $\mathcal{M} = \{-1, +1\} = \mathbb{Z}_2$. This is always so if in the vacuum the symmetry is completely broken. More generaly, if the symmetry group of the action G is broken by the vacuum to a smaller group G', then the symmetry group of the resulting vacuum will be the quotient group, $\mathcal{M} = G/G'$. We will not study this phenomenon any further, and just note in passing that a version of this mechanism is believed to be operative during the mass generation mechanism in the standard model of elementary particles and interactions, also known as the BEH mechanism, after Brout, Englert and Higgs. But let us get back to our example. Upon inserting the potential (60) into (59) and taking for simplicity $\phi_0 = 0$, we get,

$$\int_0^{\phi_\mp}\frac{d\phi'}{\phi'^2 - v_0^2} = \pm\sqrt{\frac{\lambda}{2}}(x - x_0) \Rightarrow \ln\Big(\frac{|\phi_\mp - v_0|}{\phi_\pm + v_0}\Big) = \sqrt{2}\mu(x - x_0)\,, \tag{61}$$

where μ is a mass parameter (indeed, the mass of the field in the vacuum is given by, $m^2(\phi_\pm) = (d^2V/d\phi^2)(\phi = \phi_\pm) = 2\lambda v_0^2 = 2\mu^2$). Assuming that $\phi_\pm \leq v_0$, Eq. (61) can be easily solved for $\phi_\pm = \phi_\pm(x)$,

$$\phi_\pm(x) = \pm\frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu}{\sqrt{2}}(x-x_0)\right). \tag{62}$$

These solutions is known as the static *kink*s of the theory, and unlike the vacuum solution, they carry a non-vanishing energy. Indeed, as can be easily shown by plugging in (62) into (55) (with $D = 2$), the kinks (62) carry an energy density (the energy per unit length) of

$$\rho_\pm(x) = \frac{\mu^4}{2\lambda}\frac{1}{\cosh^4\left[\frac{\mu}{\sqrt{2}}(x-x_0)\right]}\,, \qquad E[\phi_\pm] = \int_{-\infty}^{\infty} dx\rho_\pm(x) \tag{63}$$

This can be integrated to yield the total mass (or energy) of a static kink configuration,

$$E[\phi_\pm] \equiv M_{\rm kink} = \frac{2\sqrt{2}\mu^3}{3\lambda}\,. \tag{64}$$

Since the theory is Lorentz covariant, a moving kink can be obtained simply by boosting the static solution,

$$\phi_{\pm,v}(t,x) = \pm\frac{\mu}{\sqrt{\lambda}}\tanh\left(\frac{\mu}{\sqrt{2}}\gamma(x-x_0-vt)\right), \qquad \gamma = \frac{1}{\sqrt{1-v^2/c^2}}\,. \tag{65}$$

When $c > v > 0$ the kink is moving in the positive x direction; when $-c < v < 0$ it is moving in the negative x direction. Notice that, due to the nonlinearity of the theory, a superposition of two kinks (moving at different speeds centered at different positions) is not in general a solution. This is to be contrasted with electromagnetic waves, whose equations of motion are linear in the fields, and which hence satisfy a superposition principle. This superposition principle allowed us to construct a general solution to the wave equation, which could be written as a superposition of individual waves of a wave number $\vec{k}$ and a frequency $\omega = \|\vec{k}\|/c$ (for each $\vec{k}$ there is one wave with arbitrary amplitude and with a phase velocity in the direction of $\vec{k}$, and another in the opposite direction). In spite of this lack of superposition principle, linear superposition of kinks are approximate solutions to the theory, provided the separation between individual kinks is large enough. Namely, if one considers two kinks (i,j) whose centers are located at x_i and x_j, respectively, and which are moving at the speed v_i and v_j then a two kink solution will be an approximate solution provided $[\gamma_i(x-x_i-v_it)]-[\gamma_j(x-x_j-v_jt)] \gg 1/\mu$. If this is satisfied, the overlap of two kinks will be exponentially small, such that the superposition of two kinks is to a good approximation a solution of the theory. There is a special class of theories for which a superposition of certain classes of non-vacuum classical solutions is a solution of the theory. These class of theories have special form of interactions that make them (classically) integrable.
A simple calculation yields that the energy density of a moving kink is γ times larger than the energy density of a static kink, such that

$$E[\phi_{\pm,v}] = \gamma M_{\rm kink}\,. \tag{66}$$

We have thus arrived at an important result: a moving kink obeys the relativistic Einstein's enegy-mass relation for point particles. Upon some thought, this should not come as a surprise; after all we are solving a Lorentz-covariant theory. Further important remark concerns the non-perturbative character of the kink solutions (62). Namely, in the limit when $\lambda \to 0$ these solutions become meaningless (ill defined). That means that they cannot be obtained by perturbing (expanding in powers of λ) around the $\lambda = 0$ solution. This has important implications for constructing general solutions of quantum field theories which contain non-perturbative classical solutions such as the kink: perturbative expansion will completely miss these type of solutions, and they have to be considered/added separately. Such solutions go under the name of instantons, and their discussion is beyond the scope of these lectures.

We still have not discussed one important question, and that is the question of stability of classical solutions. Namely, we know that the kinks have an energy that is larger than the energy of the vacuum, so it is only natural to wonder whether and under what circumstances the kinks will decay into the true vacuum. We shall consider this question on the example of the kink of the two dimensional theory considered above.

In order to study this question, notice first that there is a conserved current associated with the moving kink (62)

$$k^\mu_\pm = \frac{\sqrt{\lambda}}{\mu}\epsilon^{\mu\nu}\partial_\nu \phi_{\pm,v}\,, \tag{67}$$

where $\epsilon^{\mu\nu}$ is the totally antisymmetric tensor, $\epsilon^{01} = 1 = -\epsilon^{10}$, $\epsilon^{00} = 0 = \epsilon^{11}$. Indeed, due to the assymetry of $\epsilon^{\mu\nu}$ it immediately follows that $\partial_\mu k^\mu_\pm = 0$. Now, let us define the charge associated with k^μ as usual,

$$Q = \int_{-\infty}^{\infty} dx k^0_\pm = \frac{\sqrt{\lambda}}{\mu}[\phi_{\pm,v}(t,\infty) - \phi_{\pm,v}(t,-\infty)] = \pm 2 \equiv Q_\pm \tag{68}$$

This charge is known as the *topological index*, and it is conserved by the evolution, namely $(d/dt)Q = 0$ (this can be easily shown by noting that $(d/dt)Q = \int dx \partial_\mu k^\mu = 0$). Hence, since the topological charge of a kink $\neq 0$, and it is conserved, it cannot be converted by the evolution to a vacuum state, which has a different topological index $Q_{\rm vac} = 0$. One can show that this conservation of topological charge also holds in quantum theory.

We have thus arrived at an important conclusion: once a (moving) kink is created, it will travel forever without being destroyed. That presents an explanation of the phenomenon initially observed an notified in scientific literature by Scott Russell. Of course, the Scott Russell's solitary wave eventually (after several kilometers) disappeared (dissipated). But that is not suprising, since the dynamics of water waves, which is described by the Navier-Stokes equation, does not embody the same type of non-linearity as the scalar theory we considered here.

The last question we shall discuss here is why is the charge Q in (68) called topological index, *i.e.* how is it – if at all – related to topology. In order to understand that, note that Q is in fact completely determined by the field value at the boundaries at $x \to \pm\infty$, which is a zero dimensional space consisting of two points, $x \in \{+\infty, -\infty\}$. (More generally, the boundary of a $(D-1)$-dimensional space is a $(D-2)$-dimensional closed hypersurface.) Since the kink represents a solution that maps one boundary onto one vacuum ($\phi_\pm(+\infty) = \pm v_0$), and another boundary onto another ($\phi_\pm(-\infty) = \mp v_0$), the topology of this mapping is said to be non-trivial. Mathematically speaking, when the 0-th homotopy group π_0 of the vacuum $\mathcal{M}$ is nontrivial, one can have topologically non-trivial solutions (such as kinks) that are stable and have a nontrivial topological index. π_0 measures the number of disconnected componets in the vacuum of the theory, such that in the case under consideration, $\pi_0(\mathcal{M}) = \pi_0(\mathbb{Z}_2) = 2$, which is nontrivial (*i.e.* different from 1).

In more complicated theories one can construct more complicated classical solutions, important examples are (global and gauged) monopoles. Gauged monopoles were first constructed independently by Gerard 't Hooft (from Utrecht) and Alexander Polyakov.