Magnetic monopoles

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Symmetry breaking

Topological solitons

GUT monopoles

The Dirac monopole

A ’t Hooft-Polyakov monopole

Observational bounds

Conclusion

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Symmetry breaking (very briefly)

- Thoroughly discussed in previous talks
- Spontaneous symmetry breaking occurs when a system with some symmetry (described by a symmetry group G) possesses vacuum states that are not invariant under this symmetry.
- Perturbations are made around one such a solution.
- Best explained through an example.

Symmetry breaking: An example

- We can for instance consider the Lagrangian:

  (in $(2+1)$D, so $x^\mu = (t, x^1, x^2)$)

$$\mathcal{L} = -\partial_\mu \phi \overline{\partial^\mu \phi} - V(\phi), \quad \text{with } V(\phi) = \lambda(1 - |\phi|^2)^2$$

where ϕ is a complex scalar field and $\lambda > 0$.
We have a U(1) symmetry under $\phi \mapsto e^{i\alpha}\phi$.

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- The energy for this system is

$$E = \int \mathrm{d}^3 x \left(|\dot{\phi}|^2 + |\nabla \phi|^2 + V(\phi) \right)$$

- $V(\phi)$ is minimal on the 'vacuum manifold' $\mathcal{M} = S^1$, so this is minimised by the constant solution

$$\phi(x, t) = \phi \in \mathcal{M}$$

Symmetry breaking: Phase transitions

- $V(\phi)$ should actually be replaced by an effective potential $V_{\text{eff}}(\phi)$ even at $T = 0$ due to loop diagrams.
- At finite temperatures this changes into $V_{\text{eff}}(\phi, T)$

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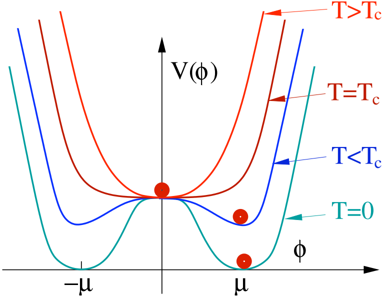
- At finite temperatures this changes into $V_{\text{eff}}(\phi, T)$
- At large temperatures the broken symmetry may be restored.



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Topological solitons

- A topological soliton is a solution that cannot be continuously deformed into the vacuum solution due to some topological constraint (the exact definition varies).
- The constraint we put on our solutions is that their total energy is finite

Topological solitons

- We had energy density (still in $2+1$ dimensions)

$$\mathcal{E} = |\dot{\phi}|^2 + |\nabla\phi|^2 + V(\phi),$$

so we need $r\,|\nabla\phi| \to 0$ and $r\,V(\phi) \to 0$ as $r \to \infty$.

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- This tells us that $\phi(r,\theta) \to \phi_\infty(\theta) \in \mathcal{M}$ as $r \to \infty$, which defines a function

$$\phi_\infty : S^1 \to \mathcal{M}, \theta \mapsto \phi(\infty, \theta).$$

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- We also need that $r\,\mathbf{e}_\theta \cdot \nabla\phi = \partial_\theta\phi \to 0$ as $r \to \infty$, so $\phi_\infty(\theta) = \phi_\infty$ actually has to be constant

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- We also need that $r\,\mathbf{e}_\theta \cdot \nabla\phi = \partial_\theta\phi \to 0$ as $r \to \infty$, so $\phi_\infty(\theta) = \phi_\infty$ actually has to be constant
- We can continuously deform such a solution to $\phi(r,\theta) = \phi_\infty$ everywhere, so there are no topological solitons (according to this definition).

Topological solitons: Gauge theory

- The same definitions also apply to gauge theories, so suppose we add some gauge field A_μ (in the usual manner) to make the symmetry local:

$$\mathcal{L} = -D_\mu\phi\overline{D^\mu\phi} - V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

- If we choose a gauge such that $A_0 = 0$ and $A_r = 0$ for $r \geq 1$, then the energy density is

$$\mathcal{E} = (\partial_0 A_i)^2 + |\partial_0\phi|^2 + |D_i\phi|^2 + V(\phi) + \frac{1}{4}(F_{ij})^2,$$

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- We still need $\lim_{r\to\infty} r\,V(\phi) = 0$, but we now require

$$\lim_{r\to\infty} r\,\mathbf{e}_\theta \cdot \mathbf{D}\phi = \lim_{r\to\infty} \partial_\theta\phi - r\,i\,e\,A_\theta\phi \to 0.$$

It is possible to *choose* an A_i such that this holds.

Topological solitons: Gauge theory

- A goes like r^{-1}, so F goes like r^{-2} and F^2 like r^{-4}.
- The first time derivatives actually forms a separate boundary value problem, so we can find a solution with finite energy:

$$\int \mathrm{d}^2x \left\{ (\partial_0 A_i)^2 + |\partial_0 \phi|^2 + |D_i \phi|^2 + V(\phi) + \tfrac{1}{4}(F_{ij})^2 \right\} < \infty$$

- The same argument also works in three spatial dimensions (but not four!) and for other fields.

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- We haven't specified ϕ, only that $\lim_{r\to\infty} r\,V(\phi) = 0$
- Any two functions with the same behaviour at infinity can be continuously transformed into each other, so only ϕ_∞ is important to classify solutions.

Topological solitons: Gauge theory

- We're interested in classes of solutions that cannot be continuously deformed into a vacuum solution.
  This comes down to classes of functions $\phi_\infty : S^{d-1}_\infty \to \mathcal{M}$ that cannot be deformed into a constant function.
- But when is this possible?

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- There is a word for continuous deformation: *homotopy*.
- We have the so-called *homotopy groups*

$$\pi_n(M) = \{f : S^n \to M\} / \sim$$

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  which exactly describe our classification.
- Our theory admits topological solitonsof this type if and only if $\pi_{d-1}(\mathcal{M}) \neq \{1\}$.

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Topological solitons: Gauge theory

- If $d = 3$ then such solutions are called *monopoles* and we can see why from the 2-dimensional case:

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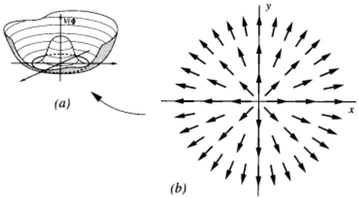
- Our example had $\mathcal{M} = S^1$ and $\pi_1(S^1) = \mathbb{Z} \neq \{1\}$, so it admits topological solitons of this type.
- We can for instance find a non-trivial solution that points radially outwards:



GUT monopoles

- GUTs embed $G_{\text{SM}} = \text{SU}(2)_{I_w} \times \text{SU}(3)_c \times \text{U}(1)_Y$ in a larger, more pleasing, compact connected gauge group G_{GUT}.

  (e.g. $G_{\text{GUT}} = \text{SU}(5)$ or $G_{\text{GUT}} = \text{SO}(10)$, etc)
- The standard model is recovered after spontaneous symmetry breaking.
- This happens after a phase transition at the GUT scale, so at around $T = T_{\text{GUT}} \sim 10^{16}$ GeV.
- Symmetry breaking in stages also possible:

$$G_{\text{GUT}} \to \ldots \to G_{\text{SM}} \to \text{SU}(3)_c \times \text{U}(1)_{\text{em}}$$

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- Do GUTs predict the existence of monopoles?

GUT monopoles: Homotopy theorems

- Assume a symmetry G is broken to H.
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- Most GUTs have (a covering group with) $\pi_2(G) = \pi_1(G) = \{1\}$, so $\pi_2(G/H) \simeq \pi_1(H)$.

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- The fundamental group of G_{SM} is

$$\begin{aligned}\pi_1(G_{\text{SM}}) &= \pi_1(\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)) \\ &= \pi_1(\text{SU}(3)) \times \pi_1(\text{SU}(2)) \times \pi_1(\text{U}(1)) \\ &= \pi_1(\text{U}(1)) = \pi_1(S^1) = \mathbb{Z}\end{aligned}$$

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- Therefore $\pi_2(\mathcal{M}) \simeq \pi_2(G_{\text{GUT}}/G_{\text{SM}}) \simeq \pi_2(G_{\text{SM}}) \simeq \mathbb{Z}$

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- At the intersection point of domains there is some probability ($p \sim 0.1$) that monopoles will form.
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- Monopole density can be estimated to be $n_{\text{MM}} \sim p\xi^{-3}$
- By causality $\xi < \ell_{\text{GUT}} \sim 10^{-27}$ cm
- This gives $n \sim 10^{80}$ cm^{-3} at the phase transition

GUT monopoles

- Annihilation can reduce this density, but not significantly, so $n_{\mathrm{MM}} \propto a^{-3}$ (and $\frac{\mathrm{d}}{\mathrm{d}t} n_{\mathrm{MM}} = -3\,H\,n_{\mathrm{MM}}$).
- The Entropy density s scales in the same way, so n/s is approximately conserved (without inflation).
- Monopole density today would therefore be $n_{\mathrm{MM,now}} = n_{\mathrm{MM,\,GUT}}(s_{\mathrm{now}})/(s_{\mathrm{GUT}})$, with $s \sim g_* T^3$

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- This gives us approximately $n_{\text{now}} \sim 10^{-7}\ \text{cm}^{-3}$, which is absurd (comparable to the baryon density). This is the *monopole problem*.
- Inflation solves this problem: As long as the temperature after preheating is below the GUT scale the monopole density will be too low to oberve.

The Dirac monopole

- Classical Magnetic monopoles with $\mathbf{B}_{\rm mm} = \frac{g}{4\,\pi\,r^2}\hat{\mathbf{x}}$ is allowed by Maxwell's equations (after extension).

$$\nabla\cdot\mathbf{E} = 4\,\pi\,\rho_{\rm e} \qquad \nabla\times E = -\frac{\partial\mathbf{B}}{\partial t} - 4\,\pi\mathbf{j}_{\rm m}$$

$$\nabla\cdot\mathbf{B} = 4\,\pi\,\rho_{\rm m} \qquad \nabla\times B = \frac{\partial\mathbf{E}}{\partial t} + 4\,\pi\mathbf{j}_{\rm e}$$

- Charge density: $\nabla\cdot\mathbf{B}_{\rm mm}(\mathbf{x}) = g\,\delta^3(\mathbf{x}) = 4\,\pi\,\rho_{\rm m}(\mathbf{x})$
- Net magnetic flux: $\int_S \mathbf{B}_{\rm mm}\cdot{\rm d}\mathbf{S} = g$

  (for any surface S around the monopole)

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- To formulate quantum mechanics, we need $\mathbf{B} = \nabla \times \mathbf{A}$.
- No such potential can be defined for the magnetic monopole, even if the origin is excluded from its domain since by Stokes' theorem:

$$\int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{A} \cdot d\mathbf{l} = 0$$

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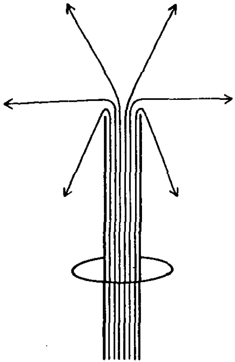
- We can define an $\mathbf{A}$ such that $\mathbf{B} = \nabla \times \mathbf{A}$ on any contractible region that does not contain the origin.

The Dirac monopole

- If we cut out a line, the magnetic field lines can disappear through it and there is no problem.
- We can define a potential $\mathbf{A}$ everywhere except in this line (so for $0 \leq \theta < \pi$)

$$\mathbf{A}^{(1)} = \frac{g}{4\pi r} \frac{1 - \cos\theta}{\sin\theta} \mathbf{e}_\varphi$$

such that $\mathbf{B}_{\text{mm}} = \nabla \times \mathbf{A}$.

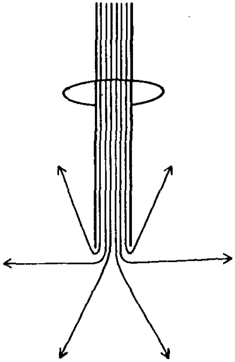


The Dirac monopole

- If we cut out a line, the magnetic field lines can disappear through it and there is no problem.
- We can define a potential **A** everywhere except in this line (so for $0 < \theta \leq \pi$)

$$\mathbf{A}^{(2)} = \frac{-g}{4\pi r}\frac{1+\cos\theta}{\sin\theta}\mathbf{e}_{\varphi}$$

such that $\mathbf{B}_{\text{mm}} = \nabla \times \mathbf{A}$.



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The Dirac monopole

- We had found $\mathbf{A}^{(1)} = \dfrac{g}{4\pi r}\dfrac{1-\cos\theta}{\sin\theta}\mathbf{e}_\varphi \qquad (0 \leq \theta < \pi)$

  and also $\mathbf{A}^{(2)} = \dfrac{-g}{4\pi r}\dfrac{1+\cos\theta}{\sin\theta}\mathbf{e}_\varphi \qquad (0 < \theta \leq \pi).$
- These are related by the gauge transformation

$$\mathbf{A}^{(2)} = \mathbf{A}^{(1)} - \nabla\alpha \qquad \text{with } \alpha = \frac{g}{2\pi}\varphi.$$

- Find a similar gauge transformation wherever you choose your 'Dirac string'.

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- Find a similar gauge transformation wherever you choose your 'Dirac string'.
- One problem: $\alpha(\varphi = 2\pi) - \alpha(\varphi = 0) = g$,
  i.e. α is multiply-valued (up to multiples of g).

The Dirac monopole

- α itself is not observable, so is this really a problem?
- A field ψ with charge e couples to $\mathbf{A}$ via

$$D_j\psi = \partial_j\varphi - i\,e\,A_j\psi$$

and it transforms under gauge transformations as

$$\psi(\mathbf{x}) \mapsto e^{-i\,e\,\alpha(\mathbf{x})}\psi(\mathbf{x})$$

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- Because α is multiply-defined we want

$$e^{-i\,e\,\alpha(\mathbf{x})}\psi(\mathbf{x}) = e^{-i\,e\,(\alpha(\mathbf{x})+g)}\psi(\mathbf{x}),$$

for which we need $e\,g \in 2\,\pi\,\mathbb{Z}$.

The Dirac monopole: Charge quantisation

- If $e\,g \in 2\,\pi\,\mathbb{Z}$ then everything is consistent.
- The monopole charge therefore needs to be a multiple of $2\,\pi/e$.

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- The monopole charge therefore needs to be a multiple of $2\,\pi/e$.
- Conversely, if a single magnetic monopole with charge g exists, then the electric charge of any particle has to be an integer multiple of $2\,\pi/g$ (charge quantisation).
- Attempts to make a quantum field theory with Dirac monopoles have failed.

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A 't Hooft-Polyakov monopole

- Consider the Georgi-Glashow SU(2) theory with the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(D_\mu \Phi D^\mu \Phi) - \frac{1}{4}\lambda(\eta^2 - |\Phi|^2)^2 - \frac{1}{4}\,\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}),$$

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- Ingredients:
  - A Higgs field $\Phi = \Phi^a t^a$
  - A gauge field $A_\mu = A_\mu^a t^a$
  - $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\,e\,[A_\mu, A_\nu]$
  - $D_\mu \Phi = \partial_\mu \Phi - i\,e\,[A_\mu, \Phi]$
- Here $t^a = \frac{1}{2}\,\tau^a$ with $[t^a, t^b] = i\,\varepsilon_{abc} t^c$ generate of SU(2)
  (τ^1, τ^2 and τ^3 are the Pauli matrices.)
  The field Φ transforms in the adjoint representation:
  $\Phi \mapsto g\,\Phi\,g^{-1}$ or $\Phi \mapsto [\xi^a t^a, \Phi]$ (infinitesimally)

A 't Hooft-Polyakov monopole

- $V(\Phi) = \frac{1}{4}\lambda(\eta^2 - |\Phi|^2)^2$ is minimised at $|\Phi| = \eta$, so we have a spontaneously broken symmetry with vacuum manifold S^2.

A 't Hooft-Polyakov monopole

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- What we usually do:
  In the unitary gauge we can write $\Phi = (\eta + \phi)t^3$ and $A_\mu = W^1_\mu t^1 + W^2_\mu t^2 + a_\mu t^3$
- The Higgs field (ϕ) gets a mass $\sqrt{2\,\lambda}\,\eta$, the W-bosons get mass $2\,\eta$ and the photon a_μ is massless.
- The unbroken U(1) is generated by t^3.

A ’t Hooft-Polyakov monopole

- A static solution to the field equations of the form

$$\Phi(x) = \eta h(r)\frac{x^a}{r}t^a, \quad A_i = \frac{1}{e}(1-k(r))\varepsilon_{ija}\frac{x^j}{r^2}t^a, \quad A_0 = 0$$

with $h(0) = k(\infty) = 0$ and $h(\infty) = k(0) = 1$ $(A_0 = 0)$
exists and has been computed numerically.

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A 't Hooft-Polyakov monopole

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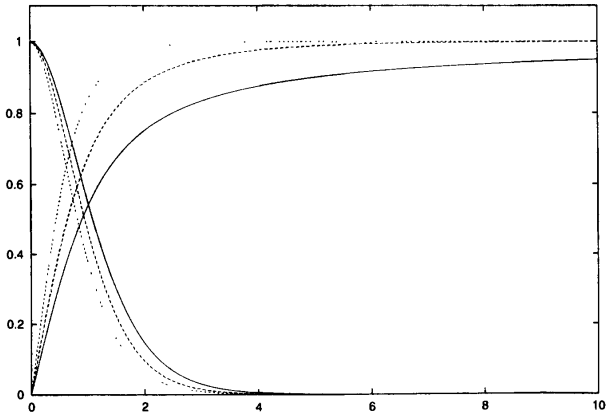
with $h(0) = k(\infty) = 0$ and $h(\infty) = k(0) = 1$ $(A_0 = 0)$
exists and has been computed numerically.

- Core size is of order $(e\,\eta)^{-1} \simeq \sqrt{\frac{137}{4\,\pi}}\,\eta^{-1}$ due to exponential convergence.
- For $r \gg (e\,\eta)^{-1}$ we get

$$\Phi(x) \simeq \eta\, \frac{x^a}{r} t^a \in S^2 \qquad \text{and} \qquad A_i \simeq \frac{1}{e}\, \varepsilon_{ija} \frac{x^j}{r^2} t^a$$

- The solution is stable and the mass (energy) can be calculated to be of order $8\,\pi\eta/e^2 \simeq 2 \times 137\,\eta$.

A 't Hooft-Polyakov monopole



h $(h(0) = 0)$ and k $(k(0) = 1)$ for $\lambda \to 0$ (solid), $\lambda = e^2/40$ (dashed) and $\lambda = e^2/4$ (dotted). The horizontal axis is scaled by $2/(e\,\eta)$.

A 't Hooft-Polyakov monopole

- We can choose $f_{\mu\nu} = 2\,\mathrm{Tr}(F_{\mu\nu}\Phi/|\Phi|) = \sum_a F^a_{\mu\nu}\frac{\Phi^a}{|\Phi|}$ and define the magnetic field as

$$B_i \equiv -\tfrac{1}{2}\varepsilon_{ijk}f_{jk} \simeq \frac{1}{e}\,\frac{x^i}{r^3} = \frac{\mathbf{r}}{e\,r^2} \qquad (r \gg (e\,\eta)^{-1})$$

- This describes a magnetic monople of charge $4\,\pi/e$

A 't Hooft-Polyakov monopole

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- This describes a magnetic monopole of charge $4\,\pi/e$

- We *can* choose the unitary gauge ($\Phi = (1+\phi)t^3$) on a contractible region that does not contain the origin.
- In this gauge we can write $F_{\mu\nu} = F^1_{\mu\nu}t^1 + F^2_{\mu\nu}t^2 + f_{\mu\nu}t^3$ and $A_\mu = W^1_\mu + W^2_\mu t^2 + a_\mu t^3$ and obtain

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

- The field equations tell us that now $\partial^\mu f_{\mu\nu} = 0$.

Symmetry breaking

Topological solitons

GUT monopoles

The Dirac monopole

A ’t Hooft-Polyakov monopole

Observational bounds

Conclusion

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Observational bounds: Overclosure

- The Friedmann equation $H^2 = \frac{8\,\pi\,G_N}{3\,c^2}\rho + \frac{\Lambda}{3} - \frac{c^2\,k}{a^2}$ can be rewritten as

$$\rho_{cr} = \rho_b + \rho_\gamma + \ldots + \rho_\Lambda + \rho_{MM} - c^2\,k/H_0^2,$$

so we need $\rho_{MM} < \rho_{cr} = \frac{3\,c^2}{8\,\pi\,G_N}H_0^2 \simeq 5 \times 10^{-5}$ GeV/cm^3 to prevent overclosure of our universe.
- Thus $n_{MM} < 5 \times 10^{-23}(m/10^{17}\ \text{GeV})^{-1}\ \text{cm}^{-3}$
- Magnetic monopoles expected to have velocity $\sim 10^{-3}c$
- Resulting flux limit:
$F < F_u \sim 10^{-15}(m_{MM}/10^{17}\ \text{GeV})^{-1}\ \text{cm}^{-2}\text{s}^{-1}\text{sr}^{-1}$.

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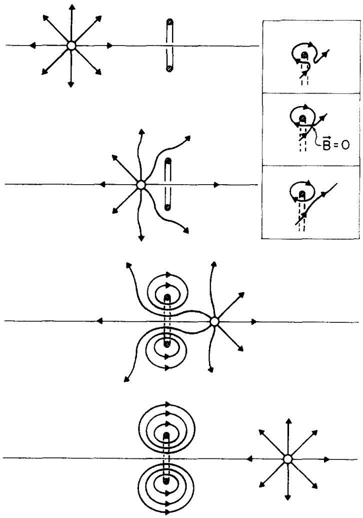
Observational bounds: The Parker bound

- Our Galaxy has a magnetic field of the order of $\sim 3\mu\text{G} \simeq 10^{-9}$ T varying over typical distances of $L \sim 10^{21}$ cm $\simeq 300$ pc.
- This is generated by the *dynamo effect* and refreshed over a typical timescale of $\tau \sim 10^8$ yrs.
- Magnetic monopoles are accelerated by these magnetic fields over a distance L and gain energy.
- The energy of the magnetic field should not be trained faster than τ
- Parker bound $F < F_\text{P} \sim 10^{-15}$ cm^{-2}s^{-1}sr^{-1} obtained
- Result was later improved:

$$F < F_\text{PE} \sim 10^{-16} (m/10^{17}\text{ GeV})\text{ cm}^{-2}\text{s}^{-1}\text{sr}^{-1}$$

Observational bounds: Induction



- A magnetic monopole travelling through a superconducting loop leaves behind a flux $2\,\Phi_0$ (flux quantum).
- Flux can be measured using SQUIDs
- Independent of velocity, mass etc.
- Shielding is a problem
- No monopoles found but upper bound obtained: $2\times10^{-14}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}\mathrm{sr}^{-1}$

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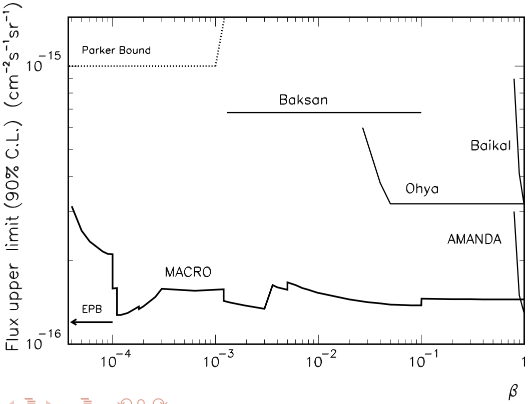
Observational bounds: Energy loss

- Fast ($v > 10^{-2}c$) magnetic monopoles are strongly ionizing
- Due to their large mass they are nevertheless very penetrating
- Exact energy loss rate depends on the monopole velocity and the material used



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Observational bounds: Catalysed nucleon decay

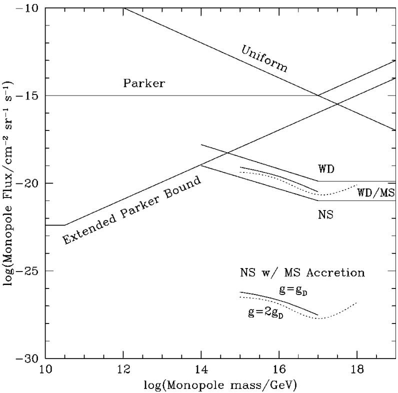
- A GUT may allow for the following

$$M + p \to M + e^+ + \pi^0, \qquad M + p \to M + \mu^+ + K^0,$$
$$M + n \to M + e^+ + \pi^-, \qquad M + n \to M + \mu^+ + K^-.$$

- Monopoles might be able to catalyse such reactions (Rubakov-Callan mechanism)
- Cross-section **may** be of order $10^{-25}/(v/10^{-3})$ cm^2
- If this is the case then this would be measurable.
- No catalysed nucleon decay has been been measured, which limits the flux (**under the above assumption**)
$F < 3 \times 10^{-16}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}\mathrm{sr}^{-1} \quad (10^{-4} < \mathrm{v} < 5 \times 10^{-3}),$
$F < 3 \times 10^{-15}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}\mathrm{sr}^{-1} \quad (10^{-5} < \mathrm{v} < 10^{-1}),$
$F < 3 \times 10^{-13}\ \mathrm{cm}^{-2}\mathrm{s}^{-1}\mathrm{sr}^{-1} \quad (10^{-2} < \mathrm{v} < 1)$

Observational bounds: Catalysed nucleon decay

- Neutron stars and white dwarfs capture all monopoles that hit them and these accumulate in the core
- **Under the conditions from the previous slide** monopoles will catalyse nucleon decay and cause these objects to heat up.
- Comparison with measured luminosity bounds flux:



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Conclusion

- A monopole in linear gauge theory can be defined as a topological soliton and the existence of monopoles is a topological property of the vacuum manifold.
- Grand Unified Theories generally predict the allow the existence of massive magnetic monopoles.
- The monopole problem doesn't need to be a problem.
- We've studied at an example of a magnetic monopole
- People are still searching, but no monopoles have thus far been observed.
  If monopoles exist they must therefore be very rare.

Questions

Any questions?