Cosmology Formulae Midterm 2013-14

Metric compatibility, covariant derivative and the (Levi-Cività) connection $\Gamma^\alpha_{\mu\nu}$:

$$\nabla_\alpha g_{\mu\nu} = 0\,,\ \nabla_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\mu_{\alpha\beta}\omega_\mu\,,\ \nabla_\alpha A^\mu = \partial_\alpha A^\mu + \Gamma^\mu_{\alpha\beta}A^\beta\,,\ \Gamma^\alpha_{\mu\nu} = g^{\alpha\beta}\Big(\partial_{(\mu}g_{\nu)\beta} - \frac{1}{2}\partial_\beta g_{\mu\nu}\Big) \tag{1}$$

Geodesic equation and geodesic deviation (λ is an affine parameter):

$$\frac{Du^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0\,,\ u^\alpha = \frac{dx^\alpha}{d\lambda}\,, \qquad \frac{D}{d\lambda}\frac{d\xi^\mu}{d\lambda} = \mathcal{R}^\mu_{\alpha\beta\gamma}u^\alpha u^\beta \xi^\gamma \tag{2}$$

Einstein's equations:

$$G_{\mu\nu} - \frac{\Lambda}{c^2}g_{\mu\nu} = \frac{8\pi G_N}{c^4}T_{\mu\nu}\,, \qquad G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} \tag{3}$$

Riemann and Ricci tensors (for a symmetric – or Levi-Cività – connection):

$$\mathcal{R}^\alpha_{\ \beta\gamma\delta} = 2\partial_{[\gamma}\Gamma^\alpha_{\delta]\beta} + 2\Gamma^\alpha_{\mu[\gamma}\Gamma^\mu_{\ \delta]\beta}\,, \quad \mathcal{R}_{\alpha\beta} = \mathcal{R}^\gamma_{\ \alpha\gamma\beta}\,, \quad \mathcal{R} = g^{\alpha\beta}\mathcal{R}_{\alpha\beta} \tag{4}$$

Covariant actions (Hilbert-Einstein and matter):

$$S_{\rm HE} = -\frac{c^4}{16\pi G_N}\int d^4x\sqrt{-g}\left(\mathcal{R} + 2\frac{\Lambda}{c^2}\right)\,, \quad S_{\rm matter} = \int d^4x\sqrt{-g}\mathcal{L}_{\rm matter}(\psi_{\rm matter}, g_{\alpha\beta}) \tag{5}$$

Einstein's equation and matter field equations are obtained by the variation principles:

$$\frac{\delta(S_{\rm HE} + S_{\rm matter})}{\delta g^{\mu\nu}} = 0\,, \quad \frac{\delta S_{\rm matter}}{\delta \psi_{\rm matter}} = 0. \tag{6}$$

Matter actions (scalar, vector, fermionic) $[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \nabla_\mu\psi = (\partial_\mu - \Gamma_\mu)\psi]$:

$$\mathcal{L}_\phi = \frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} - V(\phi) - \frac{\xi}{2}\mathcal{R}\phi^2\,, \quad \mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\,, \quad \mathcal{L}_\psi = \bar\psi \imath\gamma^\mu\nabla_\mu\psi - m_\psi\bar\psi\psi \tag{7}$$

NB: Spin(or) connection Γ_μ is determined by the compatibility condition ($\gamma^\mu = e^\mu_{\ a}\gamma^a$):

$$\nabla_\mu\gamma_\nu = \partial_\mu\gamma_\nu - \Gamma^\alpha_{\mu\nu}\gamma_\alpha - \Gamma_\mu\gamma_\nu + \gamma_\nu\Gamma_\mu = 0 \tag{8}$$

Stress-energy tensor (general and perfect fluid):

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{\mu\nu}}\,, \quad (T_{\mu\nu})_{\rm perf.\,fluid} = (\rho + \mathcal{P})\frac{u_\mu u_\nu}{c^2} - g_{\mu\nu}\mathcal{P} \tag{9}$$

Gravitational dilatation and redshift (cosmological redshift: $z(t) = (a_0/a(t)) - 1$):

$$\frac{\delta t_1}{\delta t_2} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} \simeq 1 + \frac{\phi_N(r_2) - \phi_N(r_1)}{c^2}\,, \quad \frac{E_1}{E_2} = \frac{\nu_1}{\nu_2} = \frac{\lambda_2}{\lambda_1} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} \simeq 1 + \frac{\phi_N(r_2) - \phi_N(r_1)}{c^2} \tag{10}$$

Light deflection:

$$\vec\alpha = -\frac{2}{c^2}\int_{\rm source}^{\rm observer} d\lambda \nabla_\perp \phi_N \tag{11}$$

FLRW metric (conformal time: $d\eta = dt/a$, Gauss' curvature $R_c = 1/\sqrt{|\kappa|}$):

$$ds^2 = c^2dt^2 - a^2(t)\Big(\frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\Big) \tag{12}$$

FLRW (Friedmann) equations ($H = (d/dt)\ln(a)$, $a(t_0) = a_0 = 1$):

$$H^2 = \frac{8\pi G_N}{3c^2}\rho + \frac{\Lambda}{3} - \frac{c^2\kappa}{a^2}\,, \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G_N}{3c^2}(\rho + 3\mathcal{P}) + \frac{\Lambda}{3} \tag{13}$$

These come together with a conservation equation, and another form of the 2nd equation:

$$\dot\rho + 3H(\rho + \mathcal{P}) = 0\,, \quad \dot H = -\frac{4\pi G_N}{c^2}(\rho + \mathcal{P}) + \frac{c^2\kappa}{a^2} \tag{14}$$

The principal slow roll parameter ϵ and the EoS parameter w:

$$\epsilon = -\frac{\dot H}{H^2} = \frac{d}{dt}\left(\frac{1}{H}\right)\,, \quad \text{when } \Lambda = 0 = \kappa:\ \epsilon = \frac{3}{2}(1+w)\,, \quad w = \frac{\mathcal{P}}{\rho} \tag{15}$$

(Physical) particle horizon ($ds = 0$) [comoving horizon $\ell_c = \ell_{\rm phys}/a$]:

$$\ell_{\rm phys} = \int_{r_{\rm in}}^{r} \sqrt{g_{rr}(r')}dr' = a(t)\int_{r_{\rm in}}^{r}\frac{dr'}{\sqrt{1-\kappa r'^2}} = ac(\eta - \eta_{\rm in}) \tag{16}$$

NB: $r_{\rm in}$ = initial radius (may be zero); Hubble radius: $R_H = c/H$, Hubble time: $t_H = 1/H$.

Friedmann equation and relative densities Ω_i for today ($t = t_0$, $H_0 = H(t_0)$):

$$1 = \sum_i \Omega_i + \Omega_\Lambda + \Omega_\kappa\,, \quad \Omega_i = \frac{\rho_i}{\rho_{\rm cr}}\,, \quad \rho_{\rm cr} = \frac{3c^2}{8\pi G_N}H_0^2\,, \quad \Omega_\Lambda = \frac{\Lambda}{3H_0^2}\,, \quad \Omega_\kappa = -\frac{c^2\kappa}{H_0^2} \tag{17}$$

The age of the Universe (in conformal time: multiply the integrand by $1/\tilde a$):

$$tH_0 = \int_0^a \frac{d\tilde a}{\sqrt{\Omega_{\rm m}\tilde a^{-1} + \Omega_\gamma \tilde a^{-2} + \Omega_\Lambda \tilde a^2 + \Omega_\kappa + \Omega_Q \tilde a^{-1-3w_Q}}}\,, \tag{18}$$

Apparent and absolute magnitudes; luminosity distance d_L:

$$m - M = 5\log_{10}\left(\frac{r}{\rm Mpc}\right) + 25\,, \quad \mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2} \tag{19}$$

Luminosity distance in various geometries:

$$d_L(z) = (1+z)R_c\sinh\Big(\frac{c}{H_0R_c}\int_0^z\frac{dz'}{E(z')}\Big) \qquad \text{(open universe)} \tag{20}$$

$$d_L(z) = (1+z)\frac{c}{H_0}\int_0^z\frac{dz'}{E(z')} \qquad \text{(flat universe)} \tag{21}$$

$$d_L(z) = (1+z)R_c\sin\Big(\frac{c}{H_0R_c}\int_0^z\frac{dz'}{E(z')}\Big) \qquad \text{(closed universe)} \tag{22}$$

with (here Ω_i are defined today at $t = t_0$)

$$E(z)^2 = \Omega_{\rm m}(1+z)^3 + \Omega_\gamma(1+z)^4 + \Omega_\Lambda + \Omega_Q(1+z)^{3(1+w_Q)} + \Omega_\kappa(1+z)^2\,. \tag{23}$$