

1. Introduction

States in quantum mechanics in Schrödinger picture evolve as

$$|\Psi t\rangle = \hat{U}(t, t_0)|\Psi t_0\rangle, \quad \hat{U}(t, t_0) = T \exp\left(-\frac{i}{\hbar} \int_{t_0}^t dt' \hat{H}(t')\right), \quad (1)$$

where $\hat{U}(t, t_0)$ denotes the evolution operator,

$$\hat{U}(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^t \hat{H}(t') dt' + \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^t dt' \int_{t_0}^{t'} dt'' \hat{H}(t') \hat{H}(t'') + \dots \quad (2)$$

$\hat{H}(t')$ is the Hamiltonian operator and T stands for time ordering (since $t' > t''$ the Hamiltonian operator $\hat{H}(t')$ in (2) at a later time t' is always to the left of the hamiltonial operator $\hat{H}(t'')$ at an earlier time t''). The time ordered expression (2) is known as the Dyson series. As the Hamiltonian depends on the fields, when expectation value is taken, the Feynman propagator naturally occurs when one is interested in a quantum mechanical evolution. The Feynman (or time ordered) propagator is not causal, which also means that the resulting states are not causal. But that is not a problem, since states in quantum mechanics are not directly observable. As argued in Homework 8, the quantum mechanical evolution of observable quantities, such as expectation values of hermitean operators, are always expressible in terms of causal Green functions, such as the Pauli-Jordan function, or the spectral function. (*Bra* states evolve according to an anti-time ordered exponential of the Hamiltonian, which are then expressible in terms of the anti-Feynman (or anti-time ordered) propagator.)

In these notes, we shall show how to construct the Feynman propagator for a real massless scalar field ϕ , whose action is given by ¹,

$$S[\phi] = \int d^4x \left(\frac{1}{2} (\partial_\mu \phi)(\partial_\nu \phi) \eta^{\mu\nu} \right). \quad (3)$$

The Feynman (or time ordered) propagator $\Delta_F \equiv \Delta^{++}$ for this free (*i.e.* non-interacting) theory is defined in terms of two point functions as

$$\Delta_F(x; x') = \theta(t - t') \Delta^+(x; x') + \theta(t' - t) \Delta^-(x; x') \quad (4)$$

where $\Delta^+(x; x') \equiv \Delta^{-+}(x; x')$ and $\Delta^-(x; x') \equiv \Delta^{+-}(x; x')$ are the positive and negative frequency Wightman functions, respectively, defined by

$$i\Delta^{-+}(x; x') = \langle \hat{\phi}(x) \hat{\phi}(x') \rangle, \quad i\Delta^{+-}(x; x') = \langle \hat{\phi}(x') \hat{\phi}(x) \rangle, \quad (5)$$

where, for an operator $\hat{O}(t)$, $\langle \hat{O}(t) \rangle = \text{Tr}[\hat{\rho}(t) \hat{O}(t)]$, where $\hat{\rho}(t)$ denotes the density operator. When one is dealing with a pure state $|\Psi\rangle$, then $\hat{\rho}(t) = |\Psi\rangle\langle\Psi|$, and the averaging simplifies to $\langle \hat{O}(t) \rangle \rightarrow \langle \Psi | \hat{O}(t) | \Psi \rangle$.

¹In these notes we use $c = 1$

Here I have used a more common notation, according to which the Green functions G 's for a scalar field are denoted by Δ 's. The field operators in (5) obey a Klein-Gordon equation,

$$-\partial^2 \hat{\phi}(x) = 0, \quad \partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu, \quad (6)$$

and a canonical quantization reads,

$$[\hat{\phi}(\vec{x}, t), \hat{\pi}(\vec{x}', t)] = i\hbar \delta^3(\vec{x} - \vec{x}'), \quad (7)$$

where the canonical momentum of ϕ is $\pi \equiv \partial\mathcal{L}/\partial\dot{\phi} = \partial_t\phi$.

2. Mode decomposition

The field $\hat{\phi}$ can be decomposed into modes by making use of a Fourier transform,

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3} \left(e^{i\vec{k}\cdot\vec{x}} \phi(k, t) \hat{a}_{\vec{k}} + e^{-i\vec{k}\cdot\vec{x}} \phi^*(k, t) \hat{a}_{\vec{k}}^\dagger \right) \quad (8)$$

where $\hat{a}_{\vec{k}}^\dagger$ and $\hat{a}_{\vec{k}}$ are the usual creation and annihilation operators (for each \vec{k} there is precisely one pair), you are familiar with from quantum mechanics, and which obey the following commutation relations,

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}'), \quad [\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}] = 0 = [\hat{a}_{\vec{k}}^\dagger, \hat{a}_{\vec{k}'}^\dagger]. \quad (9)$$

The annihilation operators annihilate the vacuum state $|0\rangle$ of the theory,

$$\hat{a}_{\vec{k}}|0\rangle = 0 \quad (10)$$

while the creation operator $\hat{a}_{\vec{k}}^\dagger$ creates a particle of momentum \vec{k} according to,

$$\hat{a}_{\vec{k}}^\dagger|0\rangle = |1_{\vec{k}}\rangle. \quad (11)$$

Of course, the mode functions $\phi(k, t)$ and $\phi^*(k, t)$ obey ($c = 1$),

$$\frac{d^2}{dt^2} \phi(k, t) + k^2 \phi(k, t) = 0, \quad \frac{d^2}{dt^2} \phi^*(k, t) + k^2 \phi^*(k, t) = 0.$$

The solution that is consistent with relativistic invariance and with the commutation relations (7) and (9) is of the form (can you show it?),

$$\phi(k, t) = \sqrt{\frac{\hbar}{2k}} e^{-ikt}, \quad \phi^*(k, t) = \sqrt{\frac{\hbar}{2k}} e^{ikt}. \quad (12)$$

These functions obey a canonical Wronskian, defined by,

$$W[\phi, \phi^*] \equiv \phi \dot{\phi}^* - \dot{\phi} \phi^* = i, \quad (13)$$

and represent the vacuum solutions for the problem at hand. While $\phi(k, t)$ corresponds to the positive frequency mode function, $\phi^*(k, t)$ is the negative frequency mode function. Note that their amplitude is not independent. Indeed, they are related by complex conjugation. This

relation is a consequence of reality of the scalar field, which implies $\hat{\phi}^\dagger(x) = \hat{\phi}(x)$. One can quite easily construct more general (non-vacuum solutions), but we will not go into it here. Let us now show how to evaluate the positive and negative frequency Wightman functions in the vacuum state $|0\rangle$ (in which the mode functions are given by (12)). Observe first that

$$\begin{aligned} \langle 0|\hat{a}_{\vec{k}}\hat{a}_{\vec{k}'}|0\rangle &= 0, & \langle 0|\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{k}'}^\dagger|0\rangle &= 0 \\ \langle 0|\hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{k}'}|0\rangle &= 0, & \langle 0|\hat{a}_{\vec{k}}\hat{a}_{\vec{k}'}^\dagger|0\rangle &= \langle 0|\{[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] + \hat{a}_{\vec{k}}^\dagger\hat{a}_{\vec{k}'}\}|0\rangle = (2\pi)^3\delta^3(\vec{k} - \vec{k}'), \end{aligned} \quad (14)$$

where we made use of (9) and (10). We are now ready to evaluate the Wightman functions (5), where the expectation value will be taken with respect to the vacuum state $|0\rangle$. Inserting the mode decomposition (8), and making use of (14), the Wightman functions become,

$$\begin{aligned} \imath\Delta^{-+}(x; x') &= \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \phi(k, t) \phi^*(k, t') \\ \imath\Delta^{+-}(x; x') &= \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{x}')} \phi^*(k, t) \phi(k, t'). \end{aligned} \quad (15)$$

The sign of the exponential in the Wigner transform (which is defined as a Fourier transform with respect to the relative coordinate $\vec{x} - \vec{x}'$) in the above expression tells us that $\Delta^{-+} \equiv \Delta^+$ picks the positive frequency (positive pole) contribution, while $\Delta^{+-} \equiv \Delta^-$ picks the negative frequency (negative pole) contribution. We shall perform the momentum integration in (15) in spherical coordinates for $\vec{k} : (k, \theta, \phi)$. In order to perform the angular integrations in (15), it is convenient to choose $\vec{x} - \vec{x}' = \|\vec{x} - \vec{x}'\|(0, 0, 1)$ to point in the \hat{z} -direction, such that $\vec{k} \cdot (\vec{x} - \vec{x}') = kr \cos(\theta)$, where $r = \|\vec{x} - \vec{x}'\|$. The result is,

$$\begin{aligned} \imath\Delta^{-+}(x; x') &= \frac{\hbar}{2\pi^2} \int_0^\infty dk k^2 \frac{\sin(kr)}{kr} \frac{1}{2k} e^{-ik(t-t')} = \frac{\hbar}{4\pi^2} \frac{1}{r} \int_0^\infty dk \sin(kr) e^{-ik(t-t')} \\ \imath\Delta^{+-}(x; x') &= \frac{\hbar}{4\pi^2} \frac{1}{r} \int_0^\infty dk \sin(kr) e^{ik(t-t')}, \end{aligned} \quad (16)$$

where we made use of the vacuum mode functions (12). Both of these integrals contain integrands which are complex oscillatory functions whose amplitude does not decrease as k increases, and hence the integrals are not convergent. A way to evaluate the integrals is to promote $t - t'$ to a complex number (this procedure is called analytic continuation). It suffices to add a small imaginary part to $t - t'$ according to

$$\begin{aligned} \imath\Delta^{-+}(x; x') &= \frac{\hbar}{4\pi^2} \frac{1}{r} \int_0^\infty dk \frac{e^{ikr} - e^{-ikr}}{2i} e^{-ik(t-t'-i\epsilon)} \\ \imath\Delta^{+-}(x; x') &= \frac{\hbar}{4\pi^2} \frac{1}{r} \int_0^\infty dk \frac{e^{ikr} - e^{-ikr}}{2i} e^{ik(t-t'+i\epsilon)}, \end{aligned} \quad (17)$$

where $\epsilon > 0$ is an infinitesimal positive real number. The new integrals are convergent. The physical Wightman functions are obtained by *analytic continuation* after integration, which amounts to sending $\epsilon \rightarrow 0$. It is, however, customary to keep ϵ in the expressions for the resulting Green functions, in order to serve as a reminder from what analytic continuation they originated.

The first integral in (17) yields the following positive frequency Wightman function,

$$\begin{aligned} \imath\Delta^{-+}(x; x') &= \frac{\hbar}{4\pi^2} \frac{1}{2r\imath} \left(\frac{-1}{\imath[r - (t - t' - i\epsilon)]} - \frac{-1}{-\imath[r + (t - t' - i\epsilon)]} \right) \\ &= -\frac{\hbar}{4\pi^2} \frac{1}{(t - t' - i\epsilon)^2 - r^2} \equiv -\frac{\hbar}{4\pi^2} \frac{1}{\Delta x_{-+}^2}, \end{aligned} \quad (18)$$

while the second integral yields the following positive frequency Wightman function,

$${}_i\Delta^{+-}(x; x') = -\frac{\hbar}{4\pi^2} \frac{1}{(t-t'+i\epsilon)^2 - r^2} \equiv -\frac{1}{4\pi^2} \frac{\hbar}{\Delta x_{+-}^2}, \quad (19)$$

where we defined complex ‘distance’ functions,

$$\Delta x_{+-}^2 = (t-t'+i\epsilon)^2 - \|\vec{x} - \vec{x}'\|^2, \quad \Delta x_{-+}^2 = (t-t'-i\epsilon)^2 - \|\vec{x} - \vec{x}'\|^2. \quad (20)$$

One can get rid off the ϵ parameters in (18–19) completely if one uses the Dirac identity for distributions, which states that the following two distributions are identical,

$$\frac{1}{x+i\epsilon} = \mathcal{P}\frac{1}{x} - i\pi\delta(x). \quad (21)$$

where \mathcal{P} means a principal value, *i.e.* when $1/x$ acts on a function of x the integral has to be evaluated such that the pole at $x = 0$ is approached uniformly (equally fast) from negative and positive x 's. Eqs. (18–19) can be brought into the form (21) by noting that $\Delta x_{\pm\mp}^2 = \Delta x^2 \pm \text{sign}(t-t')i\epsilon$, where $\Delta x^2 = (t-t')^2 - \|\vec{x} - \vec{x}'\|^2$ and $\text{sign}(t-t') = \theta(t-t') - \theta(t'-t)$. Thus we have,

$${}_i\Delta^{+-}(x; x') = -\mathcal{P}\frac{\hbar}{4\pi^2} \frac{1}{\Delta x^2} + i\frac{\hbar}{4\pi}\delta(\Delta x^2), \quad {}_i\Delta^{-+}(x; x') = -\mathcal{P}\frac{\hbar}{4\pi^2} \frac{1}{\Delta x^2} - i\frac{\hbar}{4\pi}\delta(\Delta x^2), \quad (22)$$

which indeed contain no more dependence on ϵ .

When (18–19) are inserted into the definition of the Feynman propagator (4) one immediately obtains the Feynman propagator for a real massless scalar field,

$$\Delta_F(x; x') = -\frac{\hbar}{4\pi^2} \frac{1}{(|t-t'| - i\epsilon)^2 - r^2} \equiv -\frac{1}{4\pi^2} \frac{\hbar}{\Delta x_{++}^2}. \quad (23)$$

Analogously, one can define the anti-time order (or anti-Feynman) propagator as,

$$\Delta_{\bar{F}}(x; x') \equiv \Delta^{--}(x; x') = \theta(t-t')\Delta^-(x; x') + \theta(t'-t)\Delta^+(x; x'). \quad (24)$$

Above considerations immediately tell us that, for a massless real scalar field, $\Delta_{\bar{F}}$ evaluates to,

$${}_i\Delta_{\bar{F}}(x; x') = -\frac{\hbar}{4\pi^2} \frac{1}{(|t-t'| + i\epsilon)^2 - r^2} \equiv -\frac{1}{4\pi^2} \frac{\hbar}{\Delta x_{--}^2}. \quad (25)$$

where we introduced a notation,

$$\Delta x_{++}^2 = (|t-t'| - i\epsilon)^2 - \|\vec{x} - \vec{x}'\|^2, \quad \Delta x_{--}^2 = (|t-t'| + i\epsilon)^2 - \|\vec{x} - \vec{x}'\|^2. \quad (26)$$

One more important remark follows. From the definition of the Feynman propagator (4–5), the canonical commutator (7) and the Klein-Gordon equation (6) one can quite easily show that the Feynman propagator obeys the following equations,

$$-\partial^2{}_i\Delta_F(x; x') = -\partial'^2{}_i\Delta_F(x; x') = i\hbar\delta^4(x-x'), \quad (27)$$

where $\partial'^2 = \eta^{\mu\nu}(\partial/\partial x'^\mu)(\partial/\partial x'^\nu)$.

3. Contour integration

Now, these expressions for the vacuum Green functions of a massless real scalar field can be also obtained without referring to mode functions, but instead working from the momentum space propagator, and performing appropriate contour integrations. Including only the pole contributions results in vacuum propagators of free theory (not including interactions). We shall now show how it works in some detail.

The propagator equation (27) together with translational symmetry of the vacuum imply that the Feynman propagator $\Delta_F(x; x') = \Delta_F(x - x')$ is a function of coordinate difference only. This allows one to perform a Wigner transform, defined by,

$$\Delta_F(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-x')} \tilde{\Delta}_F(k^\mu), \quad \tilde{\Delta}_F(k^\mu) = \int \frac{d^4k}{(2\pi)^4} e^{+ik \cdot (x-x')} \Delta_F(x - x'), \quad (28)$$

where $k \cdot (x - x') = \eta_{\mu\nu} k^\mu (x^\nu - x'^\nu)$. The propagator equation (27) in Wigner space becomes simple, $k^\mu k_\mu \tilde{\Delta}_F(k^\alpha) = \hbar$, and it is solved by

$$\tilde{G}_F(k^\mu) = \frac{\hbar}{k^\mu k_\mu} = \frac{\hbar}{2k} \left(\frac{1}{k^0 - k} - \frac{1}{k^0 + k} \right), \quad (29)$$

where $k \equiv \|\vec{k}\|$, such that in direct (physical space) the propagator becomes,

$$\Delta_F(x - x') \stackrel{?}{=} \hbar \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \int \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \frac{1}{2k} \left(\frac{1}{k^0 - k} - \frac{1}{k^0 + k} \right). \quad (30)$$

The k^0 integral is not well defined, unless one gives a prescription for how to evaluate it around the poles on the real axis at $k^0 = k$ (the positive frequency pole) and $k^0 = -k$ (the negative frequency pole). An inspection (basically trial and error!) shows that the pole prescription corresponding to the Feynman propagator is the one depicted on figure 1a. From Figure 1b we see that the integral does not change if one shifts the poles by an infinitesimal amount $\pm i\epsilon$ ($\epsilon > 0$ is an infinitesimal parameter),

$$\Delta_F(x - x') = \hbar \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{x}')} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} e^{-ik^0(t-t')} \frac{1}{2k} \left(\frac{1}{k^0 - k + i\epsilon} - \frac{1}{k^0 + k - i\epsilon} \right). \quad (31)$$

This is now a well defined expression. We shall now show how to perform the k^0 integral by contour integration. Since the Feynman propagator is defined for all times (both for t' in the past of t as well as for t' in the future of t), it is convenient to evaluate the integral separately when $t' - t > 0$ (this part is proportional to $\theta(t' - t)$), and when $t' - t < 0$ (this part is proportional to $\theta(t - t')$). The appropriate contours are shown in Figure 1c. The contours are chosen such that the semi-circular parts of the contours or radius R do not contribute in the limit when $R \rightarrow \infty$ (for a finite R the semi-circular contours give answers that are exponentially suppressed as $\sim e^{-R}$). Next, we use the Cauchy integral theorem (see *e.g.* chapters 6-7 of George Arfken, *Mathematical Methods for Physicists*) to evaluate the k^0 integrals. According to the Cauchy theorem, the k^0 integral along the real axis from $k^0 = -\infty$ to $k^0 = +\infty$ equals to the closed contour integral (if the semi-circular contour does not contribute). The two relevant contours are shown in figure 3c. The left contour gives a contribution equals to $(2\pi i)$ (since the sense of integration is positive) times the residue of the integrand at the pole $k^0 = -k + i\epsilon$, while the right contour on Figure 3c gives $(-2\pi i)$ (since

the sense of integration is negative) times the residue of the integrand at the pole $k^0 = k - i\epsilon$. The result is,

$$\Delta_F(x - x') = \hbar \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(\theta(t' - t) \frac{2\pi i}{2\pi} (-1) \frac{e^{ik(t-t')}}{2k} + \theta(t - t') \frac{-2\pi i}{2\pi} \frac{e^{-ik(t-t')}}{2k} \right). \quad (32)$$

When this is rewritten as,

$$i\Delta_F(x - x') = \hbar \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \left(\theta(t' - t) \frac{e^{ik(t-t')}}{2k} + \theta(t - t') \frac{e^{-ik(t-t')}}{2k} \right), \quad (33)$$

a comparison with Eqs. (4) and (15–16) reveals that the expressions multiplying $\theta(t' - t)$ and $\theta(t - t')$ in (33) are precisely the negative and positive frequency Wightman functions $i\Delta^{+-}$ and $i\Delta^{-+}$, respectively. In figure 1d the two contours from figure 1c have been deformed without crossing any poles. This means that the contour integration around the poles as shown in figure 1d gives the same answer as the ones in figure 1c. The contours in figure 1d represent the integration contours you should use in your homework 8 when constructing the Wightman functions.

The remaining \vec{k} integrals in (33) are performed exactly the same way as it is done in Eqs. (16–23), resulting in the Feynman propagator (23).

In these notes we have shown how to evaluate the vacuum contribution to the free Feynman propagator for a massless real scalar field. The Feynman propagator in more complicated situations such as thermal equilibrium has, in addition to the vacuum contribution originating from the quasiparticle poles $k^0 = \pm k$ we calculated in these notes, other contributions which satisfy a homogeneous wave equation (and have possibly additional poles at complex k^0). The propagator for a massive scalar field has an additional term in the Lagrangian density $\Delta\mathcal{L} = -m^2\phi^2/2$. This term changes the k^0 poles to $k^0 \equiv \pm\omega = \pm\sqrt{k^2 + m^2}$, which is the Einstein dispersion relation for massive particles. The procedure of constructing the massive Feynman propagator is in principle the same as outlined in these notes, except that now the poles are shifted, which will change the integrals. In particular, the \vec{k} integrals will be now more complicated.

The quantum mechanical Green functions can be immediately obtained from the field theoretic result, when one realises that the Feynman propagator for a quantum particle in a harmonic potential can be thought of as just one harmonic oscillator with a frequency $\omega \leftarrow k$. Hence, the quantum mechanical Green functions can be read off from our intermediate results (before the \vec{k} integrals are performed).