FINAL EXAM GR 2014: ~~FORMULAE SHEET~~

GEODESIC EQ: $\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} \equiv \frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0.$

PARALLEL TRANSPORT: $\frac{D}{d\lambda}V^\mu = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta}V^\alpha\frac{dx^\beta}{d\lambda} = 0$

LEVI-CIVITÁ CONNECTION:

$$\Gamma^\rho_{\mu\nu}(x) = \frac{1}{2}g^{\rho\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right]$$

COVARIANT DERIVATIVES:

$$D_\alpha \omega_\beta = \partial_\alpha \omega_\beta - \Gamma^\mu_{\alpha\beta}\omega_\mu \,;\, D_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^\beta_{\alpha\rho}V^\rho$$

RIEMANN TENSOR:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - (\mu \rightleftarrows \nu) \,;\, \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{(\mu\nu)}$$

KILLING EQ: $D_{(\mu}K_{\nu)} = 0$

EINSTEIN EQ: $G_{\mu\nu} + \frac{\Lambda}{c^2} = \frac{8\pi G_N}{c^4}T_{\mu\nu}$

EINSTEIN TENSOR: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$

SCALAR COVARIANT ACTION:

$$S_\phi = \int d^4x\sqrt{-g}\left\{-\frac{1}{2}(\partial_\mu\phi)(\partial_\nu\phi)g^{\mu\nu} - V(\phi)\right\} ; g = \det(g_{\mu\nu})$$

STRESS-ENERGY TENSOR: action (matter)

$$\boxed{T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\,\delta S_{\text{MATTER}} / \delta g^{\mu\nu}}$$

ELECTROMAGNETIC ACTION:

$$S_{EM} = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F_{\rho\sigma} g^{\mu\rho} g^{\nu\sigma} - j^\nu A_\nu \right\} \sqrt{-g}$$

EOM: $\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) \equiv \nabla_\mu F^{\mu\nu} = j^\nu$.

HILBERT-EINSTEIN ACTION:

$$S_{HE} = \frac{c^4}{16\pi G_N} \int d^4x \sqrt{-g} \left\{ R - \frac{2\Lambda}{c^2} \right\}$$

ENERGY CONDITIONS:

WEC: $T_{\mu\nu} t^\mu t^\nu \geq 0$; NEC: $T_{\mu\nu} l^\mu l^\nu \geq 0 \ (l^2 = 0$

DEC: WEC + $(T^{\mu\nu} t_\mu)(T_{\nu\rho} t^\rho) \leq 0$; NDEC.

SEC: $T_{\mu\nu} t^\mu t^\nu \geq -\frac{1}{2} T t^2$; $T = g^{\mu\nu} T_{\mu\nu}$.

SCHWARZSCHILD SOLUTION

$$ds^2 = -\left(1 - \frac{2G_N M}{c^2 r}\right) dt^2 + \frac{dr^2}{1 - \frac{2G_N M}{c^2 r}} + r^2 d\Omega^2$$

Killing vectors: $K = \partial_t$; $R = \partial_\phi$.

GRAVITATIONAL REDSHIFT: $\frac{\omega_2}{\omega_1} = \sqrt{\frac{g_{00}(r_1)}{g_{00}(r_2)}}$.

EDDINGTON-FINKELSTEIN COORDS:

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dv^2 + 2 dv\, dr + r^2 d\Omega^2; \quad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

EVENT HORIZONS: $H^{\pm} = \partial J^{\mp}(\mathcal{I}^{\pm})$

special case: $K = \partial_t \Rightarrow \|\partial_\mu r\|^2_H = 0$.

CHARGE: $Q = -\int d^2x \sqrt{\gamma_S}\, n_\mu s_\nu F^{\nu\mu}$; $\left(\begin{matrix} s \perp \partial\Sigma \\ n^2 = -1 \end{matrix}\right)$

(C=1) $S = \partial\Sigma$

ENERGY (MASS): (KOMAR INTEGRAL) $K = \partial_t$

$$E_R = \frac{1}{4\pi G_N} \int_{\partial\Sigma} d^2x \sqrt{\gamma_S}\, n_\mu s_\nu (D^\mu K^\nu)$$

SPIN (ANG. MOMENTUM): $R = \partial_\phi$.

$$J = -\frac{1}{8\pi G_N} \int d^2x \sqrt{\gamma_S}\, n_\mu s_\nu (D^\mu R^\nu)$$

REISSNER-NORDSTRÖM METRIC: $\boxed{c=1}$

$$ds^2 = -\Delta dt^2 + \frac{dr^2}{\Delta} + r^2 d\Omega^2; \quad \Delta = 1 - \frac{2G_N M}{r} + \frac{G_N(Q^2+P^2)}{r^2}$$

EINSTEIN'S QUADRUPOLE FORMULAS

$$\partial^2 h_{ij}^{TT} = (-\partial_0^2 + D^2) h_{ij}^{TT} = -16\pi G_N T_{ij}^{TT}$$

$$\Rightarrow h_{ij}^{TT}(\vec{x},t) = 4G_N \int \frac{d^3x'\, T_{ij}^{TT}(\vec{x}'; t_{ret})}{\|\vec{x}-\vec{x}'\|}; \quad t_{ret} = t - \|\vec{x}-\vec{x}'\|$$

$$\xrightarrow[\text{ZONE}]{\text{RADIATION}} \frac{2G_N}{\|\vec{x}\|} \frac{\partial^2}{\partial t^2} J_{ij}; \quad J_{ij} = \int d^3x'\, T_{00}(\vec{x}', t_{ret}) \begin{pmatrix} x_i' x_j' \\ -\frac{1}{3}\delta_{ij} \vec{x}'^2 \end{pmatrix}$$

HERE: $t_{ret} = t - \|\vec{x}\|$