GENERAL RELATIVITY

Homework problem set 1, due at 23.09.2016.

■ **PROBLEM 1** Constant acceleration, part II.

This problem is a continuation of Problem 5 from Tutorial problem set 2.

Consider a coordinate system moving "along" with the accelerated particle and *rigid* in the sense that distances measured with standard rods at rest in the system are constant in time (which, in the co-moving frame, is identified with the proper time of the accelerated particle).

In part I we found the transformation from the inertial system **I'**, where the accelerated system (particle) is at rest at proper time τ and coordinate time $t' = 0$. Now, we *define* a coordinate system (w^0, w, y, z) by

$$w^0 = \tau \ , \quad w = x' \ , \quad y = y' \ , \quad z = z' \ , \tag{1.1}$$

where we have set $t' = 0$. Such a coordinate system will move along with the accelerated particle. It is very special that we have defined a transformation from Minkowski spacetime to this co-moving frame. in general this is not possible. in the following we will ignore y and z.

(a) Show that the following relations between (x^0, x) and (w^0, w) hold,

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{gw^0}{c^2}\right) - 1\right] + w\cosh\left(\frac{gw^0}{c^2}\right) \ , \tag{1.2}$$

$$x^0 = \frac{c^2}{g}\sinh\left(\frac{gw^0}{c^2}\right) + w\sinh\left(\frac{gw^0}{c^2}\right) \ . \tag{1.3}$$

(w^0, w) are called Rindler coordinates (**R**).

(b) Consider two space-time points infinitesimally separated. Let them have coordinates (x^0, x) and $(x^0 + dx^0, x + dx)$. Denote the corresponding coordinates in **R** by (w^0, w) and $(w^0 + dw^0, w + dw)$. Show that

$$ds^2 \equiv dx^2 - (dx^0)^2 = dw^2 - \left(1 + \frac{gw}{c^2}\right)^2 (dw^0)^2 \tag{1.4}$$

and conclude that the Rindler coordinates are indeed rigid (lengths are constant).

(c) Show that the point with fixed coordinate w in **R** as seen from **I**, performs a hyperbolic motion with velocity

$$v = \frac{\frac{gx^0}{c}}{\sqrt{\left(1 + \frac{gw}{c^2}\right)^2 + \left(\frac{gx^0}{c^2}\right)^2}} \quad \left(= c\tanh\left(\frac{gw^0}{c^2}\right)\right) \ . \tag{1.5}$$

(d) Show that an observer at rest in **I**, with spatial coordinate x can send signals to any other observer at rest in **I**. Consider a point $w > 0$. Show that there are points x which can never send signals to any observer at rest in **R** with coordinate w. You can use results of part (a) and (b) of this exercise. Characterize the region of space-time which cannot send signals to any observer at rest in **R**. We say that the system **R** has a horizon: There are regions of space-time which can receive signals from **R**, but cannot send signals to **R**.

Hint: You may find the following relations useful

$$\cosh\big(\text{arcsinh}x\big) = \sqrt{1+x^2}\ , \qquad \tanh\big(\text{arcsinh}x\big) = \frac{x}{\sqrt{1+x^2}}\ . \tag{1.6}$$

■ **PROBLEM 2** The energy-momentum tensor.

Recall the Euler equations for an ideal fluid with density $\rho(x^i,t)$ and velocity $v^i(x^j,t)$,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v^i)}{\partial x^i} = 0\ , \tag{2.1}$$

$$\frac{\partial p^j}{\partial t} + \frac{\partial(p^j v^i)}{\partial x^i} = (\text{force density})^j = -\frac{\partial p}{\partial x^j}\ , \tag{2.2}$$

where $p^j \equiv \rho v^j$ is the j4-component of the momentum density and p is the pressure. The first equation (the continuity equation) expresses the conservation of mass, and the second equation expresses that the change of (a component of) momentum per volume according to Newton's second law is equal to the force component per volume, i.e. minus the gradient of the pressure.

Using the continuity equation in the momentum equation, the latter can be rewritten as

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1}{\rho}\nabla p\ , \tag{2.3}$$

while the continuity equation can be written as a current conservation,

$$\partial_\mu j^\mu = 0\ , \qquad j^\mu = (\rho, \rho\vec{v})\ . \tag{2.4}$$

For an ideal fluid we have

$$T^{\mu\nu} = p\eta^{\mu\nu} + (p+\rho)U^\mu U^\nu\ , \qquad \partial_\nu T^{\nu\mu} = 0\ , \tag{2.5}$$

where $U^\mu = \gamma(v)(1, v^i)$ is the four-velocity and $(U_\mu U^\mu) = -1$. For convenience, we take $c = 1$.

(a) Write out the equations explicitly for $\mu = 0$ and for $\mu = i$, and show (using both equations) that the one for $\mu = i$ can be written as

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\frac{1-v^2}{p+\rho}\left(\nabla p + \vec{v}\frac{\partial p}{\partial t}\right)\ . \tag{2.6}$$

(b) Show that this equation reduces to (2.3) in the non-relativistic limit, and that the equation for $\mu = 0$ reduces to (2.4) in the same limit. Note that you have to reintroduce c in the equations and approximate for $v \ll c$.