GENERAL RELATIVITY

Homework problem set 2, due at 07.10.2016.

■ PROBLEM 3 Light cones. (*9 points*)

Consider a 1+1 dimensional space-time, whose metric $(0,2)-$tensor is given by,

$$ds^2 = -dt \otimes dt + a(t)^2 dx \otimes dx\,. \tag{3.1}$$

where $a(t)$ is a scale factor ($a(t)dx$ is a 1-form field that can be used to measure physical distances and dt is the 1-form field that can be used to measure time lapses). Assume that

$$\begin{aligned} a(t) &= t^{1/\epsilon} \quad (t > 0) \\ a(t) &= (-t)^{1/\epsilon} \quad (t < 0)\,, \end{aligned} \tag{3.2}$$

where ϵ is a constant. (ϵ is defined as the rate of change of the inverse expansion rate, $\epsilon = (d/dt)[1/H(t)]$, where $H(t) = (d/dt)\ln[a(t)]$. Consider a vector field, $V = d/d\lambda$ ($V^\mu = dx^\mu/d\lambda$ in some coordinate system x^μ). The light cones are then defined as,

$$ds^2(V, V) = 0\,. \tag{3.3}$$

(a) (*1 point*) Show that (3.3) implies the following differential equations for the light-cones,

$$\frac{dt}{d\lambda} = \pm a(t)\frac{dx}{d\lambda}\,. \tag{3.4}$$

(b) (*4 points*) Solve this equation when $a(t)$ is given in (3.2). Show that the solution for $\epsilon \neq 1$ and $\epsilon \neq 0$ and $t > 0$ can be written as,

$$x(t) = \pm\frac{t^{1-1/\epsilon}}{1-1/\epsilon} + x_0\,. \tag{3.5}$$

Sketch how the light cones look like for $\epsilon > 1$ (decelerating expansion, $d^2a/dt^2 < 0$) and how for for $\epsilon < 1$ (accelerating expansion, $d^2a/dt^2 > 0$). You will find that, unlike in Minkowski space, there are cases in which light cones do not intersect in the future or in the past. Also, light cones sometimes stop at a finite time. Provide physical explanation of these facts.

(c) (*2 points*) Consider the special cases $\epsilon = 0$ and $\epsilon = 1$ and analyse their light cone structure. *Hint:* Show first that in the limit when $\epsilon \to 0$ ($\epsilon \to 1$) $a(t) \to \mathrm{e}^{H_0 t}$ ($a(t) \to a_0 \times (t/t_0), t > 0$), where H_0 is a constant expansion rate and t_0 is the time at which $a = 0$.

(d) (*2 points*) Consider next contracting space-times (whose scale factor is given by the $t < 0$ case in Eq. (3.2)) and analyse the corresponding light-cones for the cases when $\epsilon > 1$ and $0 < \epsilon < 1$ in the same way as above.

■ **PROBLEM 4** Prolate spheroidal coordinates. (*6 points*)

Prolate spheroidal coordinates can be used to simplify the Kepler problem in celestial mechanics. They are related to the usual Cartesian coordinates (x, y, z) of Euclidean three-sphere by,

$$\begin{aligned} x &= \sinh(\chi)\sin(\theta)\cos(\phi)\,, && (0 \le \phi < 2\pi) \\ y &= \sinh(\chi)\sin(\theta)\sin(\phi)\,, && (0 \le \theta \le \pi) \\ z &= \cosh(\chi)\cos(\theta)\,, && (-\infty < \chi < +\infty)\,. \end{aligned} \tag{4.1}$$

(a) (*2 points*) Construct the coordinate transformation matrix $\partial x^\mu/\partial\tilde{x}^\alpha$ relating (x, y, z) to (χ, θ, ϕ).

(b) (*1 point*) Write down the line element ds^2 look like in prolate spheroidal coordinates?

(c) (*3 points*) A particle moves along a parametrised curve,

$$x(\lambda) = \cos(\lambda)\,, \qquad y(\lambda) = \sin(\lambda)\,, \qquad z(\lambda) = \sqrt{2\sinh(2\lambda)}\,. \tag{4.2}$$

Express the path of the curve in the $(\chi(\lambda), \theta, \phi)$ coordinates, *i.e.* find $\chi = \chi(\lambda)$, $\theta = \theta(\lambda)$ and $\phi = \phi(\lambda)$. These functions define geodesics. They are not well defined for all values of λ. Spell out the range of λ for which the geodesics are well defined. Can the geodesics be extended, and if yes how. *Hint:* You should get the following answer:

$$\chi(\lambda) = \ln\left[\mathrm{e}^{\lambda} + \sqrt{\mathrm{e}^{2\lambda} + 1}\right], \qquad \sin[\theta(\lambda)] = \mathrm{e}^{-\lambda}\,, \qquad \phi(\lambda) = \lambda\,. \tag{4.3}$$