GENERAL RELATIVITY

Homework problem set 3, due on 21.10.2016.

■ PROBLEM 5 Derivation of the geodesic equation. *(8 points)*

Consider the action for the particle of mass m in some 4-dimensional space-time given by metric $g_{\mu\nu}(x)$,

$$S[x(\lambda)] = \int_{\lambda_1}^{\lambda_2} d\lambda\, L\Big(x(\lambda), \dot{x}(\lambda)\Big) \ , \qquad \dot{x}^\mu(\lambda) \equiv \frac{dx^\mu}{d\lambda} \ , \tag{5.1}$$

where the Lagrangian of the particle is

$$L\Big(x(\lambda), \dot{x}(\lambda)\Big) = -mc\sqrt{-g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu} \ . \tag{5.2}$$

(a) *(2 points)* Show that the action is invariant under the reparametrization of the trajectory of the particle, i.e. show that S has the same form if we define a new parametrization $\widetilde{\lambda} = f(\lambda)$ with $\frac{df}{d\lambda} > 0$, and consider a new path $\widetilde{x}(\widetilde{\lambda}) \equiv x(f^{-1}(\widetilde{\lambda}))$.

(b) *(2 points)* Show that in flat space ($g_{\mu\nu} = \eta_{\mu\nu}$) in the non-relativistic limit ($|\vec{v}| \ll c$) the action above reduces to the usual action for the non-relativistic free particle. (What about the extra constant term?)

(c) *(2 points)* Show that if we use the proper time to parametrize the trajectory ($\lambda = \tau$) then the Euler-Lagrange equations

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0 \ , \tag{5.3}$$

can be written as

$$\frac{d}{d\tau}\left(g_{\mu\nu}(x)\frac{dx^\nu}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}(x)}{\partial x^\mu}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \ . \tag{5.4}$$

Hint: Show that the (differential) definition of τ implies

$$g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2 \ . \tag{5.5}$$

(d) *(2 points)* Show that this equation of motion can be written compactly as

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \ , \tag{5.6}$$

where $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols. This is the *geodesic equation*.

■ **PROBLEM 6** Christoffel symbols and covariant derivatives II. *(6 points)*

(a) *(2 points)* Show that for a diagonal metric Christoffel symbols are $\Gamma^{\alpha}_{\mu\nu} = 0$ when $\mu \neq \nu \neq \alpha \neq \mu$.

(b) *(2 points)* D'Alembert operator in flat (Minkowski) space-time is defined as $\Box \equiv \partial_\mu \partial^\mu = \eta^{\mu\nu}\partial_\mu\partial_\nu$. A covariant generalization of this operator to curved space-times is

$$\Box \equiv \nabla_\mu \nabla^\mu = g^{\mu\nu}\nabla_\mu\nabla_\nu \ . \tag{6.1}$$

Show that the action of a d'Alembert operator on a scalar is the following

$$\Box\phi = \frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\Big) \ . \tag{6.2}$$

Hint: Show that

$$\Gamma^{\mu}_{\mu\alpha} = \frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g} \ . \tag{6.3}$$

(c) *(2 points)* Using result (6.2) calculate $\Box\phi$ for the space-time of the 2-sphere,

$$ds^2 = -c^2dt^2 + R^2\Big(d\vartheta^2 + \sin^2(\vartheta)d\varphi^2\Big) \ . \tag{6.4}$$