GENERAL RELATIVITY

Homework problem set 4, due at 04.11.2016.

■ **PROBLEM 7** Geodesics on a rotating disk. *(16 points)*

The spatial part of the metric of a rotating disk is given by

$$ds^2 = dr^2 + \frac{r^2}{1 - \frac{r^2\omega^2}{c^2}}\, d\vartheta^2 \ . \tag{7.1}$$

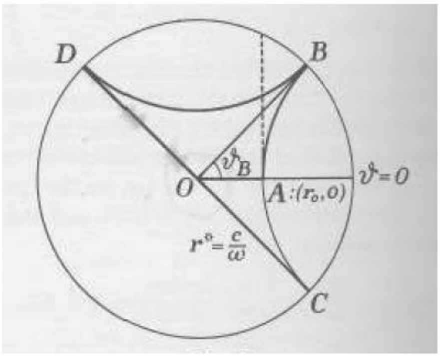

Figure 1: Geodesics on a space given by (7.1)

(a) *(3 points)* Write down the geodesic equations.

(b) *(3 points)* Consider the second-order equation for $\vartheta(s)$ (the one that involves a second derivative of $\theta(s)$). Show that the integral of motion of that equation is

$$\frac{d\vartheta}{ds} = \frac{\alpha}{r^2}\left(1 - \frac{r^2\omega^2}{c^2}\right) \ , \tag{7.2}$$

where α is a constant of integration. Next, plug in (7.2) in the second-order equation for $r(s)$, and show that it can be integrated to yield

$$\frac{dr}{ds} = \pm\sqrt{\beta - \frac{\alpha^2}{r^2}} \ , \tag{7.3}$$

where β is another constant of integration. Now, show that β is not independent of α, by imposing a constraint coming from (7.1), but

$$\beta = 1 + \frac{\alpha^2\omega^2}{c^2} \ . \tag{7.4}$$

Finally, conclude that

$$\frac{dr}{d\vartheta} = \pm\frac{r^2\sqrt{1 + \frac{\alpha^2\omega^2}{c^2} - \frac{\alpha^2}{r^2}}}{\alpha\left(1 - \frac{r^2\omega^2}{c^2}\right)} \ . \tag{7.5}$$

By integrating this expression we can in principle solve for $r(\vartheta)$.

(c) *(2 points)* Consider a geodesic passing through $(r_0, 0)$ and having $dr/ds = 0$ there. Express α in terms of r_0.

(d) *(2 points)* Find the geodesic corresponding to $\alpha = 0$.

(e) *(2 points)* Show that the geodesics always cross the boundary $r_* = c/\omega$ at a right angle (see Fig. 1).

(f) *(2 points)* Show that the angle φ between two geodesics which go through the same point, expressed in terms of α_1 and α_2 and the r-coordinate at the point where they meet, is given by

$$\cos\varphi = \pm\sqrt{1 + \frac{\alpha_1^2\omega^2}{c^2} - \frac{\alpha_1^2}{r^2}}\sqrt{1 + \frac{\alpha_2^2\omega^2}{c^2} - \frac{\alpha_2^2}{r^2}} + \frac{\alpha_1\alpha_2}{r^2}\left(1 - \frac{r^2\omega^2}{c^2}\right) . \tag{7.6}$$

(g) *(2 points)* Show that the sum of angle of the triangle OAB in the figure is less than π. How small and how large can the sum of the angles in a triangle formed by geodesics be?

■ PROBLEM 8 Spaces of constant curvature. *(6 points)*

Consider the following two 2-dimensional spaces,

$$\text{(I)} \quad ds^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\varphi^2 \ , \tag{8.1}$$

$$\text{(II)} \quad ds^2 = d\chi^2 + \frac{\sin^2(\sqrt{\kappa}\chi)}{\kappa} d\varphi^2 \, . \tag{8.2}$$

(a) *(2 points)* Find all (non-vanishing) Christoffel symbols.

(b) *(2 points)* Calculate the Riemann tensor and the Ricci tensor for both cases (I) and (II).

(c) *(2 points)* Calculate the Ricci scalars for both cases (I) and (II). For 2-dimensional spaces the Ricci scalar completely characterizes the space. Can you establish what these spaces are in the cases when (i) $\kappa > 0$, and (ii) $\kappa < 0$ and what is the relation between them?