GENERAL RELATIVITY

Homework problem set 7, due at 13.01.2017.

■ **PROBLEM 14** Reissner-Nordström black hole. *(20 points)*

In this problem you will derive the solution of the Einstein equation and the Maxwell equation for a spherically symmetric electrically charged black hole, called the Reissner-Nordström black hole. Note that even though we assume that the charge and mass are located at the center of the black hole, we must solve the Einstein equation with a source. The electric charge is a source for the electric field, this field carries energy, and is therefore a source for the Einstein equation,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \ . \tag{14.1}$$

In addition, the Maxwell equation for the electromagnetic field also has to be satisfied,

$$\nabla_\mu F^{\mu\nu} = 0 \ . \tag{14.2}$$

We start by simplifying the Einstein equation.

(a) *(2 points)* The energy-momentum tensor of the electromagnetic field is given by

$$T_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \ . \tag{14.3}$$

Show that it is traceless, and calculate the Ricci scalar using that property. One again we need not calculate the Ricci scalar explicitly.

Next we make the ansatz for the solution. As before, we assume a spherically symmetric line element,

$$ds^2 = -e^{2\alpha(r,t)} dt^2 + e^{2\beta(r,t)} dr^2 + r^2 d\Omega^2 \ . \tag{14.4}$$

(The relevant geometric quantities for this metric are given at the end of this problem.) The electric and magnetic field also have to respect spherical symmetry. That limits them to have just the radial component,

$$E_r = E_r(t,r) \ , \qquad B_r = B_r(t,r) = 0 \ , \tag{14.5}$$

where we have assumed a particular case of zero magnetic field.

(b) *(2 points)* Recall that the electric field is given as $E_i = F_{0i}$ in terms of the EM field strength tensor. Therefore, the only non-vanishing component is $F_{tr} = f(t,r)$. Determine all the components of the EM energy-momentum tensor in terms of functions f, α, β.

Now we examine the components of the Einstein equation to further constrain functions α and β, and to fnd their connection with f.

(c) *(2 points)* Show that the (tr) component of the Einstein equation implies that β is time independent, $\beta(t, r) = \beta(r)$.

(d) *(2 points)* By making an apropriate linear combination of (tt) and (rr) components of the Einstein equation show that

$$\alpha(t, r) = -\beta(r) + \gamma(t) \ , \tag{14.6}$$

and show that we can freely set $\gamma(t) = 0$, since we can absorb it in the definition of the time coordinate.

(e) *(2 points)* Write down the remaining Einstein equation relating β and f.

Finally, we have to examine the Maxwell equation.

(f) *(2 points)* Even though it could have been argued in (e) that f is independent of time, show that it follows from the r component of the Maxwell equation (14.2).

(g) *(2 points)* Show that the only remaining non-trivial component of the Maxwell equation is the t component.

(h) *(2 points)* Solve the equation obtained in (g) to find

$$f = \frac{C}{r^2} \ , \tag{14.7}$$

where C is an integration constant. We set it to $C = Q/\sqrt{4\pi}$, where Q has the interpretation of the total charge of the black hole.

(i) *(2 points)* Go back to the equation obtained in (e) and solve it to obtain

$$e^{-2\beta} = 1 - \frac{R_S}{r} + \frac{G_N Q^2}{r^2} \ . \tag{14.8}$$

Obviously, when $Q = 0$ (no electric field) this solution must reduce to the Schwarzschild one, and we must have $R_S = 2MG_N$.

(j) *(2 points)* In these coordinates the condition for the horizon(s) is

$$0 = 1 - \frac{R_S}{r} + \frac{G_N Q^2}{r^2} \ . \tag{14.9}$$

Discuss the existence of horizons depending on the ranges of M and Q of the black hole. Do the parameter choices for which there are no horizons represent physically viable solutions?

For a discussion on the different rgions of the Reissner-Nordström black hole read section 6.5 from Carroll's book.

The non-vanishing components of the Ricci tensor for metric (14.4) are

$$R_{tt} = \Big[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\Big] + e^{2(\alpha-\beta)}\Big[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta + \frac{2}{r}\partial_r\alpha\Big] \ , \tag{14.10}$$

$$R_{rr} = -\Big[\partial_r^2\alpha + (\partial_r\alpha)^2 - \partial_r\alpha\partial_r\beta - \frac{2}{r}\partial_r\beta\Big] + e^{2(\beta-\alpha)}\Big[\partial_t^2\beta + (\partial_t\beta)^2 - \partial_t\alpha\partial_t\beta\Big] \ , \tag{14.11}$$

$$R_{tr} = R_{rt} = \frac{2}{r}\partial_t\beta \ , \qquad R_{\vartheta\vartheta} = e^{-2\beta}\Big[r(\partial_r\beta - \partial_r\alpha) - 1\Big] + 1 \ , \qquad R_{\varphi\varphi} = R_{\vartheta\vartheta}\sin^2\vartheta \ , \tag{14.12}$$

and the non-vanishing Christoffel symbols for that metric are

$$\Gamma^t_{tt} = \partial_t\alpha \ , \qquad \Gamma^t_{tr} = \Gamma^t_{rt} = \partial_r\alpha \ , \qquad \Gamma^t_{rr} = e^{2(\beta-\alpha)}\partial_t\beta \ , \tag{14.13}$$

$$\Gamma^r_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha \ , \qquad \Gamma^r_{tr} = \Gamma^r_{rt} = \partial_t\beta \ , \qquad \Gamma^r_{rr} = \partial_r\beta \ , \tag{14.14}$$

$$\Gamma^\vartheta_{r\vartheta} = \Gamma^\vartheta_{\vartheta r} = \frac{1}{r} \ , \qquad \Gamma^r_{\vartheta\vartheta} = -re^{-2\beta} \ , \qquad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \ , \tag{14.15}$$

$$\Gamma^r_{\varphi\varphi} = -re^{-2\beta}\sin^2\vartheta \ , \qquad \Gamma^\vartheta_{\varphi\varphi} = -\sin\vartheta\cos\vartheta \ , \qquad \Gamma^\varphi_{\vartheta\varphi} = \Gamma^\varphi_{\varphi\vartheta} = \cot\vartheta \ . \tag{14.16}$$

■ PROBLEM 15 Friedmann-Lemaître-Robertson-Walker space-time. *(11 points)*

The most general line element which respects the assumptions of homogeneity and isotropy of the spatial part (but is not static) is

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right] \ . \tag{15.1}$$

This is the FLRW line element. Here κ is a constant which can be larger than zero, smaller than zero, or zero, which corresponds to the constant time slices being closed, open, or flat, respectively. The time-dependent function $a(t)$ is called the scale factor, and it can be thought of as the time-dependence of physical length between two comoving observers (observers at fixed spatial coordinates).

Plugging this line element into the Einstein equation with some sort of ideal fluid as a source (with energy density ρ and pressure p, and assuming the fluid rest frame), and assuming $\Lambda > 0$, gives the Friedmann equations,

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G_N}{3}\rho - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \ , \tag{15.2}$$

$$\dot H + H^2 = \frac{\ddot a}{a} = -\frac{4\pi G_N}{3}(\rho + 3p) + \frac{\Lambda}{3} \ , \tag{15.3}$$

where $H(t)$ is the Hubble parameter.

(a) *(2 points)* Consider these equations in the case when $\Lambda = 0$ and $\kappa = 0$, and the ideal fluid satisfies the linear equation of state $p = w\rho$, where $w = \text{const}$. Find the scale factor as a function of time, with initial conditions $a(t_0) = a_0$, $\rho(t_0) = \rho_0$. Show that the quantity

$$\epsilon \equiv -\frac{\dot H}{H^2} \ , \tag{15.4}$$

is a constant in the case of ideal fluids satisfying the assumed equation of state.

(b) *(3 points)* Solve the Friedmann equations assuming: (i) $\rho = p = 0$, $\kappa = 0$ and $\Lambda > 0$; (ii) $\rho = p = 0$, $\Lambda = 0$ and $\kappa > 0$, (iii) $\rho = p = 0$, $\Lambda = 0$ and $\kappa < 0$.

(c) *(2 points)* We define the *conformal time* η via $dt = a\,d\eta$, where a is the scale factor. Show that in the case $\kappa = 0$, the metric written in conformal time is conformal to the Minkowski one, i.e.

$$ds^2 = a^2(\eta)\Big[- d\eta^2 + dr^2 + r^2 d\Omega^2 \Big] , \qquad d\Omega^2 = d\theta^2 + \sin^2(\theta) d\phi^2 \ . \tag{15.5}$$

Find what the solution of (a) is in conformal time.

(d) *(4 points)* Use the approach as for Minkowski metric to define the needed transformations and draw the conformal diagram for the FLRW space-time (15.5) for $\epsilon =$ const. Treat separately the cases $\epsilon > 1$ and $0 < \epsilon < 1$.