GENERAL RELATIVITY

Problem set 1, on 09.09.2016.

■ PROBLEM 1 Problem 1.2 from [1]

Imagine that space (not spacetime) is actually a finite box, or in more sophisticated terms, a three-torus, of size L. By this, we mean that there is a coordinate system $x^\mu = (t, x, y, z)$ such that every point with coordinates (t, x, y, z) is *identified* with every point with coordinates $(t, x + L, y, z)$, $(t, x, y + L, z)$ and $(t, x, y, z + L)$. Note that the time coordinate is the same. Now consider two observers; observer A is at rest in this coordinate system (constant spatial coordinates), while observer B moves in the x-direction with constant velocity v. A and B begin at the same event, and while A remains still, B moves once around the universe and comes back to intersect the worldline of A without ever having to accelerate (since the universe is periodic). What are the relative proper times experienced in this interval by A and B? Is this consistent with your understanding of Lorentz invariance?

■ PROBLEM 2 Problem 1.4 from [1]

Projection effects can trick you into thinking that an astrophysical object is moving "superluminally". Consider a quasar that ejects gas with speed v at an angle θ with respect to the line-of sight with angular speed $\frac{v_{app}}{D}$, where D is the distance to the quasar and v_{app} is the apparent speed. Derive an expression for v_{app} in terms of v and θ. Show that, for appropriate values of v and θ, v_{app} can be greater than 1.

■ PROBLEM 3 Problem 1.5 from [1]

Particle physicists are so used to setting $c = 1$ that they measure mass in units of energy. In particular, they tend to use electron volts (1 eV $= 1.6 \times 10^{-12}$ erg $= 1.8 \times 10^{-33}$ g), or, more commonly, keV, MeV, and GeV (10^3 eV, 10^6 eV and 10^9 eV, respectively). The muon has been measured to have a mass of 0.106 GeV and a rest frame lifetime of 2.19×10^{-6} seconds. Imagine that such a muon is moving in the circular storage ring of a particle accelerator, 1 kilometer in diameter, such that the muon's total energy is 1000 GeV. How long would it appear to live from the experimenter's point of view? How many radians would it travel around the ring?

■ PROBLEM 4 Lorentz covariant formulation of electromagnetism.

The Maxwell action for electromagnetism is

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{4\pi}{c} j^\mu A_\mu \right] , \qquad (4.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, A_μ is the vector potential, and $j^\mu = (c\rho, \vec{j})$ is the electromagnetic (charge) current. Furthermore,

$$F^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix} , \tag{4.2}$$

where $\eta^{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric tensor, and $\vec{E} = (E^1, E^2, E^3)$ and $\vec{B} = (B^1, B^2, B^3)$ are the electric and magnetic fields, respectively.

(a) By making use of the variational principle, $\delta S/\delta A_\mu(x) = 0$, show that the implied equation of motion is

$$\partial_\nu F^{\nu\mu} = \frac{4\pi}{c} j^\mu . \tag{4.3}$$

Show further that (4.3) implies the usual inhomogeneous Maxwell's equations,

$$\nabla \cdot \vec{E} = 4\pi\rho , \qquad -\frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} . \tag{4.4}$$

(b) Show that the dual field tensor,

$$^*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix} , \tag{4.5}$$

obeys a homogeneous equation,

$$\partial_\nu {}^*F^{\nu\mu} = 0 . \tag{4.6}$$

Here $\epsilon^{\mu\nu\alpha\beta}$ is the so-called Levi-Civita tensor, which is defined to be

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} 1 & \text{if } (\mu\nu\alpha\beta) = \text{any even permutation of } (0123) , \\ -1 & \text{if } (\mu\nu\alpha\beta) = \text{any odd permutation of } (0123) , \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

Show further that these equations imply the homogeneous Maxwell's equations,

$$\nabla \cdot \vec{B} = 0 , \qquad \frac{1}{c}\frac{\partial}{\partial t}\vec{B} + \nabla \times \vec{E} = 0 . \tag{4.8}$$

(c) Recall that under Lorentz transformation $F^{\mu\nu}$ transforms as

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta} . \tag{4.9}$$

Consider first an ordinary rotation around the z-axis. Show that the magnetic and electric fields transform as ordinary vectors under such a rotation. Consider next a boost along the x-axis with velocity v. Show that we have the following transformation, mixing electric and magnetic fields ($\gamma = 1/\sqrt{1-\beta^2}$, $\beta = v/c$),

$$\begin{aligned} B_1' &= B_1 , \quad B_2' = \gamma(B_2 + \beta E_3) , \quad B_3' = \gamma(B_3 - \beta E_2) , \\ E_1' &= E_1 , \quad E_2' = \gamma(E_2 - \beta B_3) , \quad E_3' = \gamma(E_3 + \beta B_2) . \end{aligned} \tag{4.10}$$

Recall that a Lorentz boost along the x-axis is given by the transformation

$$x' = \gamma(x - \beta x^0) , \qquad y' = y , \qquad z' = z , \qquad x'^0 = \gamma(x^0 - \beta x) . \tag{4.11}$$

(d) Consider now an inertial system **I** and the Lorentz boosted system **I'** (where **I'** is not rotated relative to **I**), moving with velocity $\vec{v}$ relative to **I**. The formula generalizing formula (4.10) above is

$$\vec{B} = \gamma \vec{B}' + \frac{\vec{v}}{v^2}(\vec{v} \cdot \vec{B}')(1-\gamma) + \gamma \frac{\vec{v}}{c} \times \vec{E}' \ , \tag{4.12}$$

$$\vec{E} = \gamma \vec{E}' + \frac{\vec{v}}{v^2}(\vec{v} \cdot \vec{E}')(1-\gamma) - \gamma \frac{\vec{v}}{c} \times \vec{B}' \ . \tag{4.13}$$

Assume that $\vec{E}'$ and $\vec{B}'$ are constant and different from zero. Find the condition that $\vec{E}'$ and $\vec{B}'$ have to satisfy in order that there exists a $\vec{v}$ such that $\vec{E} = 0$, and find the corresponding $\vec{v}$ expressed in terms of $\vec{E}'$ and $\vec{B}'$. (Remark: by considering charged particles in this special Lorentz frame one easily finds how they will move.)

(e) Show that the two quantities $\vec{E} \cdot \vec{B}$ and $|\vec{E}|^2 - |\vec{B}|^2$ are lorentz invariant, *i.e.* they are Lorentz scalars.

Hint: You might find the following identities useful,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \ , \tag{4.14}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} \ , \tag{4.15}$$

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{B} \cdot \vec{C})(\vec{A} \cdot \vec{D}) \ . \tag{4.16}$$

Try to prove these identites by using the antisymmetric symbol, ϵ^{ijk}. Recall that $(\vec{A} \times \vec{B})^i = \epsilon^{ijk} A^j B^k$.

References

[1] Sean M. Carroll, "An introduction to General Relativity, Spacetime and Geometry" Addison Wesley (2004), ISBN 0-8053-8732-3.