GENERAL RELATIVITY

Tutorial problem set 10, 18.11.2016.

■ **PROBLEM 27** Palatini formalism.

The Einstein equation can be derived by varying the action $S = S_{EH} + S_{matter}$ with respect to the metric, where the Einstein-Hilbert action is considered a functional of the metric only. That means we have expressed the connection in terms of the metric (Christoffel connection).

The Einstein equation can also be derived by treating the metric and connection as independent degrees of freedom, and varying the action with respect to them separately. This is know as the Palatini formalism. Here, the Einstein-Hilbert action is

$$S_{EH} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\Gamma) \ , \tag{27.1}$$

and $R_{\mu\nu}$ depends on the (unspecified) connection only.

(a) Show that the variation with respect to the metric leads to the Einstein equation

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \ . \tag{27.2}$$

Note that the Einstein tensor $G_{\mu\nu}$ is defined in terms of Ricci tensor, which in turn depends on the yet unspecified connection. In order for this equation to make sense we need to vary the action with respect to the connection.

In the remainder of this problem you will show that if we assume a torsion-less connection, $\Gamma^{\alpha}_{[\mu\nu]} = 0$, that the variational principle tells us it also has to be metric compatible, $\nabla_\alpha g^{\mu\nu} = 0$. This then leads to (27.2) being the standard Einstein equation for the metric.

(b) What is the variation of the Ricci tensor with respect to the connection?

(c) Utilizing (ordinary) partial integration show that the requirement that the Einstein-Hilbert action be stationary under variation with respect to the connection implies the following relation

$$0 = -\frac{1}{\sqrt{-g}}\,\partial_\alpha\left(\sqrt{-g}g^{\mu\nu}\right) + g^{\mu\nu}\Gamma^{\sigma}_{\sigma\alpha} - g^{\sigma\nu}\Gamma^{\mu}_{\sigma\alpha} - g^{\mu\sigma}\Gamma^{\nu}_{\alpha\sigma} + \delta^{\nu}_{\alpha}\left[\frac{1}{\sqrt{-g}}\,\partial_\sigma(\sqrt{-g}g^{\mu\sigma}) + \Gamma^{\mu}_{\sigma\rho}g^{\sigma\rho}\right] \ . \tag{27.3}$$

Note that we cannot use the covariant version of Stokes theorem, since it only works for Christoffel connection, and here the connection is general.

(d) Show the following,

$$\frac{1}{\sqrt{-g}}\partial_\alpha\sqrt{-g} = -\frac{1}{2}g_{\rho\lambda}\partial_\alpha g^{\rho\lambda} \ . \tag{27.4}$$

(e) Using (27.4) and the definition of the covariant derivative, rewrite all the partial derivatives in (27.3) in terms of covariant ones, and show that the condition (27.3) reduces to

$$0 = -\nabla_\alpha g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} g_{\rho\sigma} \nabla_\alpha g^{\rho\sigma} + \delta^\nu_\alpha \left[\nabla_\lambda g^{\mu\lambda} - \frac{1}{2} g^{\mu\lambda} g_{\rho\sigma} \nabla_\lambda g^{\rho\sigma} \right] \ . \qquad (27.5)$$

(f) By appropriate contractions of (27.5) with the metric tensor show (27.5) implies

$$\nabla_\alpha g^{\mu\nu} = 0 \ , \qquad (27.6)$$

that is, the Palatini variational principle implies metric compatibility.